Strengthen Your Understanding

In Problems 24–25, explain what is wrong with the statement.

24. Let $P_n(x)$ be a Taylor approximation of degree n for a function $f(x)$ about a, where a is a constant. Then $|f(a) - P_n(a)| > 0$ for any n.

25. Let $f(x)$ be a function whose Taylor series about $x = 0$ converges to $f(x)$ for all x. Then there exists a positive integer n such that the n^{th}-degree Taylor polynomial $P_n(x)$ for $f(x)$ about $x = 0$ satisfies the inequality

$$|f(x) - P_n(x)| < 1 \quad \text{for all values of } x.$$

In Problems 26–28, give an example of:

26. A function $f(x)$ whose Taylor series converges to $f(x)$ for all values of x.

27. A polynomial $P(x)$ such that $|1/x - P(x)| < 0.1$ for all x in the interval $[1, 1.5]$.

28. A function $f(x)$ and an interval $[-c, c]$ such that the value of M in the error of the second-degree Taylor polynomial of $f(x)$ centered at 0 on the interval could be 4.

Decide if the statements in Problems 29–33 are true or false. Assume that the Taylor series for a function converges to that function. Give an explanation for your answer.

29. Let $P_n(x)$ be the n^{th} Taylor polynomial for a function f near $x = a$. Although $P_n(x)$ is a good approximation to f near $x = a$, it is not possible to have $P_n(x) = f(x)$ for all x.

30. If $|f^{(n)}(x)| < 10$ for all $n > 0$ and all x, then the Taylor series for f about $x = 0$ converges to $f(x)$ for all x.

31. If $f^{(n)}(0) \geq n!$ for all n, then the Taylor series for f near $x = 0$ diverges at $x = 0$.

32. If $f^{(n)}(0) \geq n!$ for all n, then the Taylor series for f near $x = 0$ diverges at $x = 1$.

33. If $f^{(n)}(0) \geq n!$ for all n, then the Taylor series for f near $x = 0$ diverges at $x = 1/2$.

10.5 FOURIER SERIES

We have seen how to approximate a function by a Taylor polynomial of fixed degree. Such a polynomial is usually very close to the true value of the function near one point (the point at which the Taylor polynomial is centered), but not necessarily at all close anywhere else. In other words, Taylor polynomials are good approximations of a function *locally*, but not necessarily *globally*. In this section, we take another approach: we approximate the function by trigonometric functions, called *Fourier approximations*. The resulting approximation may not be as close to the original function at some points as the Taylor polynomial. However, the Fourier approximation is, in general, close over a larger interval. In other words, a Fourier approximation can be a better approximation globally. In addition, Fourier approximations are useful even for functions that are not continuous. Unlike Taylor approximations, Fourier approximations are periodic, so they are particularly useful for approximating periodic functions.

Many processes in nature are periodic or repeating, so it makes sense to approximate them by periodic functions. For example, sound waves are made up of periodic oscillations of air molecules. Heartbeats, the movement of the lungs, and the electrical current that powers our homes are all periodic phenomena. Two of the simplest periodic functions are the square wave in Figure 10.19 and the triangular wave in Figure 10.20. Electrical engineers use the square wave as the model for the flow of electricity when a switch is repeatedly flicked on and off.

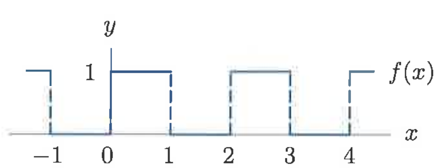


Figure 10.19: Square wave

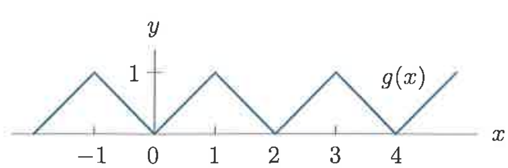


Figure 10.20: Triangular wave

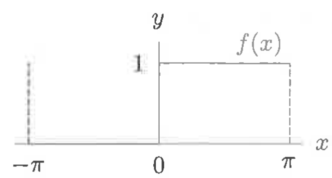
Fourier Polynomials

We can express the square wave and the triangular wave by the formulas

$$f(x) = \begin{cases} \vdots & \vdots \\ 0 & -1 \le x < 0 \\ 1 & 0 \le x < 1 \\ 0 & 1 \le x < 2 \\ 1 & 2 \le x < 3 \\ 0 & 3 \le x < 4 \\ \vdots & \vdots \end{cases} \qquad g(x) = \begin{cases} \vdots & \vdots \\ -x & -1 \le x < 0 \\ x & 0 \le x < 1 \\ 2-x & 1 \le x < 2 \\ x-2 & 2 \le x < 3 \\ 4-x & 3 \le x < 4 \\ \vdots & \vdots \end{cases}$$

However, these formulas are not particularly easy to work with. Worse, the functions are not differentiable at various points. Here we show how to approximate such functions by differentiable, periodic functions.

Since sine and cosine are the simplest periodic functions, they are the building blocks we use. Because they repeat every 2π, we assume that the function f we want to approximate repeats every 2π. (Later, we deal with the case where f has some other period.) We start by considering the square wave in Figure 10.21. Because of the periodicity of all the functions concerned, we only have to consider what happens in the course of a single period; the same behavior repeats in any other period.



$$f(x) = \begin{cases} 0 & -\pi \le x < 0 \\ 1 & 0 \le x < \pi \end{cases}$$

Figure 10.21: Square wave on $[-\pi, \pi]$

We will attempt to approximate f with a sum of trigonometric functions of the form

$$\begin{aligned} f(x) &\approx F_n(x) \\ &= a_0 + a_1 \cos x + a_2 \cos(2x) + a_3 \cos(3x) + \cdots + a_n \cos(nx) \\ &\quad + b_1 \sin x + b_2 \sin(2x) + b_3 \sin(3x) + \cdots + b_n \sin(nx) \\ &= a_0 + \sum_{k=1}^{n} a_k \cos(kx) + \sum_{k=1}^{n} b_k \sin(kx). \end{aligned}$$

$F_n(x)$ is known as a *Fourier polynomial of degree* n, named after the French mathematician Joseph Fourier (1768–1830), who was one of the first to investigate it.[5] The coefficients a_k and b_k are called *Fourier coefficients*. Since each of the component functions $\cos(kx)$ and $\sin(kx)$, $k = 1, 2, \ldots, n$, repeats every 2π, $F_n(x)$ must repeat every 2π and so is a potentially good match for $f(x)$, which also repeats every 2π. The problem is to determine values for the Fourier coefficients that achieve a close match between $f(x)$ and $F_n(x)$. We choose the following values:

The Fourier Coefficients for a Periodic Function f of Period 2π

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx, \\ a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx \quad \text{for } k > 0, \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx \quad \text{for } k > 0. \end{aligned}$$

Notice that a_0 is just the average value of f over the interval $[-\pi, \pi]$.

[5]The Fourier polynomials are not polynomials in the usual sense of the word.

For an informal justification for the use of these values, see page 573. In addition, the integrals over $[-\pi, \pi]$ for a_k and b_k can be replaced by integrals over any interval of length 2π.

Example 1 Construct successive Fourier polynomials for the square wave function f, with period 2π, given by

$$f(x) = \begin{cases} 0 & -\pi \le x < 0 \\ 1 & 0 \le x < \pi. \end{cases}$$

Solution Since a_0 is the average value of f on $[-\pi, \pi]$, we suspect from the graph of f that $a_0 = \frac{1}{2}$. We can verify this analytically:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{2\pi}\int_{0}^{\pi} 1\,dx = 0 + \frac{1}{2\pi}(\pi) = \frac{1}{2}.$$

Furthermore,

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx = \frac{1}{\pi}\int_{0}^{\pi} 1\cos x\,dx = 0$$

and

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\,dx = \frac{1}{\pi}\int_{0}^{\pi} 1\sin x\,dx = \frac{2}{\pi}.$$

Therefore, the Fourier polynomial of degree 1 is given by

$$f(x) \approx F_1(x) = \frac{1}{2} + \frac{2}{\pi}\sin x,$$

and the graphs of the function and the first Fourier approximation are shown in Figure 10.22.

We next construct the Fourier polynomial of degree 2. The coefficients a_0, a_1, b_1 are the same as before. In addition,

$$a_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(2x)\,dx = \frac{1}{\pi}\int_{0}^{\pi} 1\cos(2x)\,dx = 0$$

and

$$b_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(2x)\,dx = \frac{1}{\pi}\int_{0}^{\pi} 1\sin(2x)\,dx = 0.$$

Since $a_2 = b_2 = 0$, the Fourier polynomial of degree 2 is identical to the Fourier polynomial of degree 1. Let's look at the Fourier polynomial of degree 3:

$$a_3 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(3x)\,dx = \frac{1}{\pi}\int_{0}^{\pi} 1\cos(3x)\,dx = 0$$

and

$$b_3 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(3x)\,dx = \frac{1}{\pi}\int_{0}^{\pi} 1\sin(3x)\,dx = \frac{2}{3\pi}.$$

So the approximation is given by

$$f(x) \approx F_3(x) = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin(3x).$$

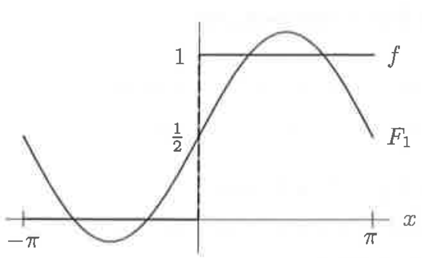


Figure 10.22: First Fourier approximation to the square wave

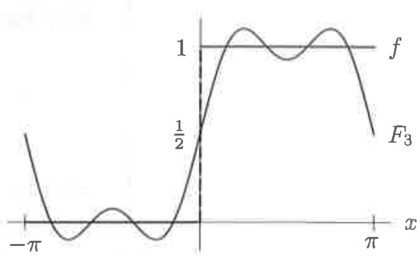


Figure 10.23: Third Fourier approximation to the square wave

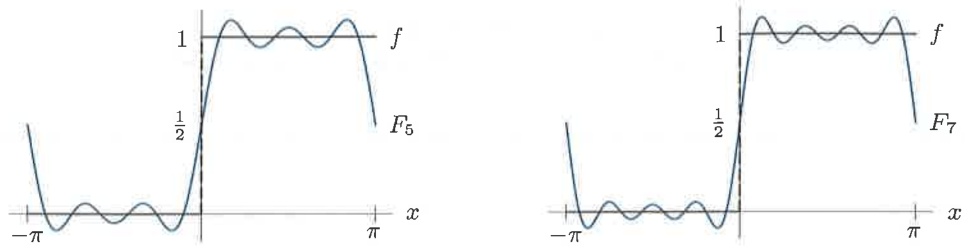


Figure 10.24: Fifth and seventh Fourier approximations to the square wave

The graph of F_3 is shown in Figure 10.23. This approximation is better than $F_1(x) = \frac{1}{2} + \frac{2}{\pi}\sin x$, as comparing Figure 10.23 to Figure 10.22 shows.

Without going through the details, we calculate the coefficients for higher-degree Fourier approximations:

$$F_5(x) = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin(3x) + \frac{2}{5\pi}\sin(5x)$$
$$F_7(x) = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin(3x) + \frac{2}{5\pi}\sin(5x) + \frac{2}{7\pi}\sin(7x).$$

Figure 10.24 shows that higher-degree approximations match the step-like nature of the square wave function more and more closely.

We could have used a Taylor series to approximate the square wave, provided we did not center the series at a point of discontinuity. Since the square wave is a constant function on each interval, all its derivatives are zero, and so its Taylor series approximations are the constant functions: 0 or 1, depending on where the Taylor series is centered. They approximate the square wave perfectly on each piece, but they do not do a good job over the whole interval of length 2π. That is what Fourier polynomials succeed in doing: they approximate a curve fairly well everywhere, rather than just near a particular point. The Fourier approximations above look a lot like square waves, so they approximate well *globally*. However, they may not give good values near points of discontinuity. (For example, near $x = 0$, they all give values near $1/2$, which are incorrect.) Thus Fourier polynomials may not be good *local* approximations.

> Taylor polynomials give good *local* approximations to a function;
> Fourier polynomials give good *global* approximations to a function.

Fourier Series

As with Taylor polynomials, the higher the degree of the Fourier approximation, generally the more accurate it is. Therefore, we carry this procedure on indefinitely by letting $n \to \infty$, and we call the resulting infinite series a *Fourier series.*

> **The Fourier Series for f on $[-\pi, \pi]$**
>
> $$f(x) = a_0 + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \cdots$$
> $$+ b_1\sin x + b_2\sin 2x + b_3\sin 3x + \cdots$$
>
> where a_k and b_k are the Fourier coefficients.

Thus, the Fourier series for the square wave is

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x + \frac{2}{7\pi}\sin 7x + \cdots.$$

Harmonics

Let us start with a function $f(x)$ that is periodic with period 2π, expanded in a Fourier series:

$$\begin{aligned} f(x) = a_0 &+ a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots \\ &+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots \end{aligned}$$

The function

$$a_k \cos kx + b_k \sin kx$$

is referred to as the k^{th} *harmonic* of f, and it is customary to say that the Fourier series expresses f in terms of its harmonics. The first harmonic, $a_1 \cos x + b_1 \sin x$, is sometimes called the *fundamental harmonic* of f.

Example 2 Find a_0 and the first four harmonics of a *pulse train* function f of period 2π shown in Figure 10.25:

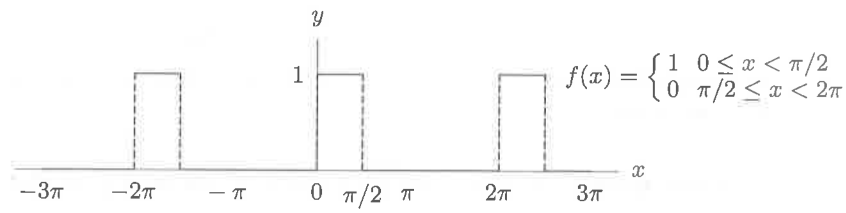


Figure 10.25: A train of pulses with period 2π

Solution First, a_0 is the average value of the function, so

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{\pi/2} 1\,dx = \frac{1}{4}.$$

Next, we compute a_k and b_k, $k = 1, 2, 3$, and 4. The formulas

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx = \frac{1}{\pi}\int_0^{\pi/2} \cos(kx)\,dx$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx = \frac{1}{\pi}\int_0^{\pi/2} \sin(kx)\,dx$$

lead to the harmonics

$$\begin{aligned} a_1 \cos x + b_1 \sin x &= \frac{1}{\pi}\cos x + \frac{1}{\pi}\sin x \\ a_2 \cos(2x) + b_2 \sin(2x) &= \frac{1}{\pi}\sin(2x) \\ a_3 \cos(3x) + b_3 \sin(3x) &= -\frac{1}{3\pi}\cos(3x) + \frac{1}{3\pi}\sin(3x) \\ a_4 \cos(4x) + b_4 \sin(4x) &= 0. \end{aligned}$$

Figure 10.26 shows the graph of the sum of a_0 and these harmonics, which is the fourth Fourier approximation of f.

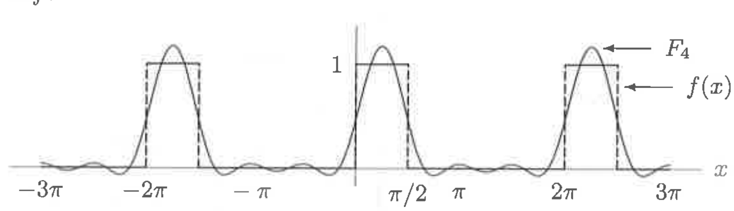


Figure 10.26: Fourth Fourier approximation to pulse train f equals the sum of a_0 and the first four harmonics

Energy and the Energy Theorem

The quantity $A_k = \sqrt{a_k^2 + b_k^2}$ is called the amplitude of the k^{th} harmonic. The square of the amplitude has a useful interpretation. Adopting terminology from the study of periodic waves, we define the *energy* E of a periodic function f of period 2π to be the number

$$E = \frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2\,dx.$$

Problem 19 on page 576 asks you to check that for all positive integers k,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} (a_k\cos(kx) + b_k\sin(kx))^2\,dx = a_k^2 + b_k^2 = A_k^2.$$

This shows that the k^{th} harmonic of f has energy A_k^2. The energy of the constant term a_0 of the Fourier series is $\frac{1}{\pi}\int_{-\pi}^{\pi} a_0^2\,dx = 2a_0^2$, so we make the definition

$$A_0 = \sqrt{2}a_0.$$

It turns out that for all reasonable periodic functions f, the energy of f equals the sum of the energies of its harmonics:

The Energy Theorem for a Periodic Function f of Period 2π

$$E = \frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2\,dx = A_0^2 + A_1^2 + A_2^2 + \cdots$$

where $A_0 = \sqrt{2}a_0$ and $A_k = \sqrt{a_k^2 + b_k^2}$ (for all integers $k \geq 1$).

The graph of A_k^2 against k is called the *energy spectrum* of f. It shows how the energy of f is distributed among its harmonics.

Example 3
(a) Graph the energy spectrum of the square wave of Example 1.
(b) What fraction of the energy of the square wave is contained in the constant term and first three harmonics of its Fourier series?

Solution
(a) We know from Example 1 that $a_0 = 1/2$, $a_k = 0$ for $k \geq 1$, $b_k = 0$ for k even, and $b_k = 2/(k\pi)$ for k odd. Thus

$$A_0^2 = 2a_0^2 = \frac{1}{2}$$

$$A_k^2 = 0 \quad \text{if } k \text{ is even}, \quad k \geq 1,$$

$$A_k^2 = \left(\frac{2}{k\pi}\right)^2 = \frac{4}{k^2\pi^2} \quad \text{if } k \text{ is odd}, \quad k \geq 1.$$

The energy spectrum is graphed in Figure 10.27. Notice that it is customary to represent the energy A_k^2 of the k^{th} harmonic by a vertical line of length A_k^2. The graph shows that the constant term and first harmonic carry most of the energy of f.

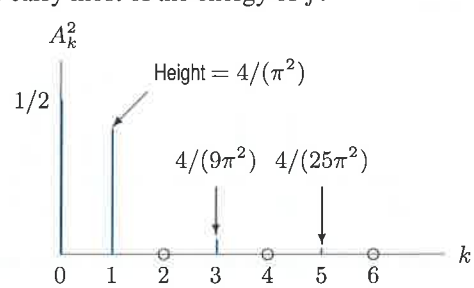


Figure 10.27: The energy spectrum of a square wave

(b) The energy of the square wave $f(x)$ is

$$E = \frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx = \frac{1}{\pi}\int_{0}^{\pi}1\,dx = 1.$$

The energy in the constant term and the first three harmonics of the Fourier series is

$$A_0^2 + A_1^2 + A_2^2 + A_3^2 = \frac{1}{2} + \frac{4}{\pi^2} + 0 + \frac{4}{9\pi^2} = 0.950.$$

The fraction of energy carried by the constant term and the first three harmonics is

$$0.95/1 = 0.95, \text{ or } 95\%.$$

Musical Instruments

You may have wondered why different musical instruments sound different, even when playing the same note. A first step might be to graph the periodic deviations from the average air pressure that form the sound waves they produce. This has been done for clarinet and trumpet in Figure 10.28.[6] However, it is more revealing to graph the energy spectra of these functions, as in Figure 10.29. The most striking difference is the relative weakness of the second and fourth harmonics for the clarinet, with the second harmonic completely absent. The trumpet sounds the second harmonic with as much energy as it does the fundamental.

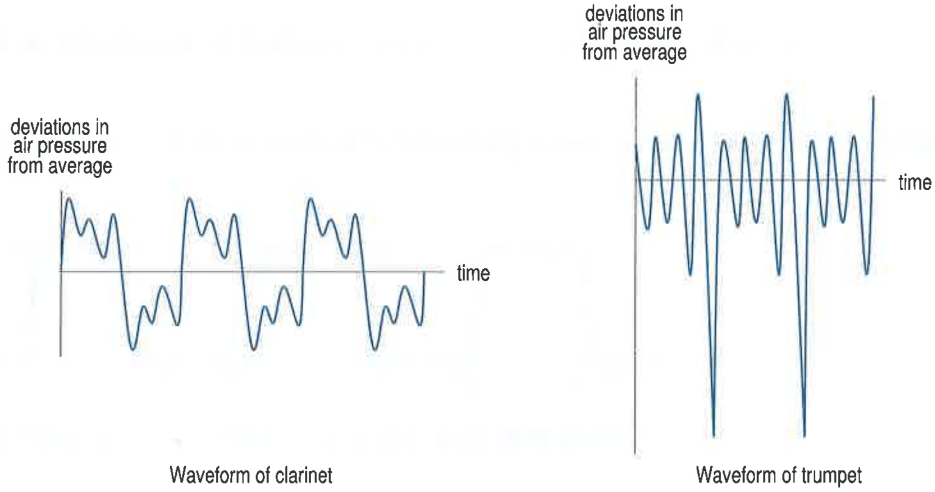


Figure 10.28: Sound waves of a clarinet and trumpet

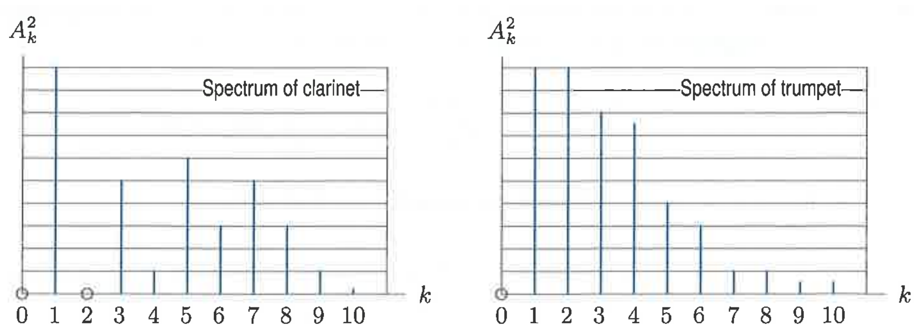


Figure 10.29: Energy spectra of a clarinet and trumpet

[6] Adapted from C.A. Culver, *Musical Acoustics* (New York: McGraw-Hill, 1956) pp. 204, 220.

What Do We Do If Our Function Does Not Have Period 2π?

We can adapt our previous work to find the Fourier series for a function of period b. Suppose $f(x)$ is given on the interval $[-b/2, b/2]$. In Problem 31, we see how to use a change of variable to get the following result:

The Fourier Series for f on $[-b/2, b/2]$

$$f(x) = a_0 + \sum_{k=1}^{\infty}\left(a_k \cos\left(\frac{2\pi kx}{b}\right) + b_k \sin\left(\frac{2\pi kx}{b}\right)\right)$$

where $a_0 = \frac{1}{b}\int_{-b/2}^{b/2} f(x)\,dx$ and, for $k \geq 1$,

$$a_k = \frac{2}{b}\int_{-b/2}^{b/2} f(x)\cos\left(\frac{2\pi kx}{b}\right)dx \quad \text{and} \quad b_k = \frac{2}{b}\int_{-b/2}^{b/2} f(x)\sin\left(\frac{2\pi kx}{b}\right)dx.$$

The constant $2\pi k/b$ is called the angular frequency of the k^{th} harmonic; b is the period of f.

Note that the integrals over $[-b/2, b/2]$ can be replaced by integrals over any interval of length b.

Example 4 Find the fifth-degree Fourier polynomial of the square wave $f(x)$ graphed in Figure 10.30.

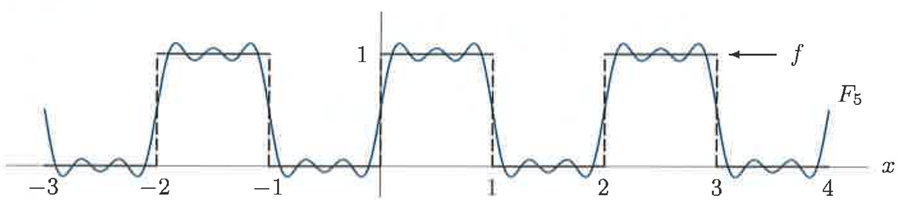


Figure 10.30: Square wave f and its fifth Fourier approximation F_5

Solution Since $f(x)$ repeats outside the interval $[-1, 1]$, we have $b = 2$. As an example of how the coefficients are computed, we find b_1. Since $f(x) = 0$ for $-1 < x < 0$,

$$b_1 = \frac{2}{2}\int_{-1}^{1} f(x)\sin\left(\frac{2\pi x}{2}\right)dx = \int_0^1 \sin(\pi x)dx = -\frac{1}{\pi}\cos(\pi x)\Big|_0^1 = \frac{2}{\pi}.$$

Finding the other coefficients by a similar method, we have

$$f(x) \approx \frac{1}{2} + \frac{2}{\pi}\sin(\pi x) + \frac{2}{3\pi}\sin(3\pi x) + \frac{2}{5\pi}\sin(5\pi x).$$

Notice that the coefficients in this series are the same as those in Example 1. This is because the graphs in Figures 10.24 and 10.30 are the same except with different scales on the x-axes.

Seasonal Variation in the Incidence of Measles

Example 5 Fourier approximations have been used to analyze the seasonal variation in the incidence of diseases. One study[7] done in Baltimore, Maryland, for the years 1901–1931, studied $I(t)$, the average number of cases of measles per 10,000 susceptible children in the t^{th} month of the year. The data points in Figure 10.31 show $f(t) = \log I(t)$. The curve in Figure 10.31 shows the second Fourier approximation of $f(t)$. Figure 10.32 contains the graphs of the first and second harmonics of $f(t)$, plotted separately as deviations about a_0, the average logarithmic incidence rate. Describe what these two harmonics tell you about incidence of measles.

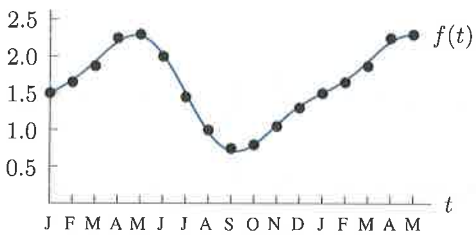


Figure 10.31: Logarithm of incidence of measles per month (dots) and second Fourier approximation (curve)

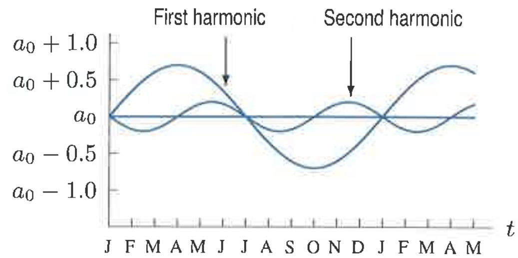


Figure 10.32: First and second harmonics of $f(t)$ plotted as deviations from average log incidence rate, a_0

Solution Taking the log of $I(t)$ has the effect of reducing the amplitude of the oscillations. However, since the log of a function increases when the function increases and decreases when it decreases, oscillations in $f(t)$ correspond to oscillations in $I(t)$.

Figure 10.32 shows that the first harmonic in the Fourier series has a period of one year (the same period as the original function); the second harmonic has a period of six months. The graph in Figure 10.32 shows that the first harmonic is approximately a sine function with amplitude about 0.7; the second harmonic is approximately the negative of a sine function with amplitude about 0.2. Thus, for t in months ($t = 0$ in January),

$$\log I(t) = f(t) \approx a_0 + 0.7\sin\left(\frac{\pi}{6}t\right) - 0.2\sin\left(\frac{\pi}{3}t\right),$$

where $\pi/6$ and $\pi/3$ are introduced to make the periods 12 and 6 months, respectively. We can estimate a_0 from the original graph of f: it is the average value of f, approximately 1.5. Thus

$$f(t) \approx 1.5 + 0.7\sin\left(\frac{\pi}{6}t\right) - 0.2\sin\left(\frac{\pi}{3}t\right).$$

Figure 10.31 shows that the second Fourier approximation of $f(t)$ is quite good. The harmonics of $f(t)$ beyond the second must be rather insignificant. This suggests that the variation in incidence in measles comes from two sources, one with a yearly cycle that is reflected in the first harmonic and one with a half-yearly cycle reflected in the second harmonic. At this point the mathematics can tell us no more; we must turn to the epidemiologists for further explanation.

Informal Justification of the Formulas for the Fourier Coefficients

Recall that the coefficients in a Taylor series (which is a good approximation locally) are found by differentiation. In contrast, the coefficients in a Fourier series (which is a good approximation globally) are found by integration.

[7]From C. I. Bliss and D. L. Blevins, *The Analysis of Seasonal Variation in Measles* (Am. J. Hyg. 70, 1959), reported by Edward Batschelet, *Introduction to Mathematics for the Life Sciences* (Springer-Verlag, Berlin, 1979).

We want to find the constants $a_0, a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ in the expression

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx).$$

Consider the integral

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \left(a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx) \right) dx.$$

Splitting the integral into separate terms, and assuming we can interchange integration and summation, we get

$$\begin{aligned} \int_{-\pi}^{\pi} f(x)\,dx &= \int_{-\pi}^{\pi} a_0\,dx + \int_{-\pi}^{\pi} \sum_{k=1}^{\infty} a_k \cos(kx)\,dx + \int_{-\pi}^{\pi} \sum_{k=1}^{\infty} b_k \sin(kx)\,dx \\ &= \int_{-\pi}^{\pi} a_0\,dx + \sum_{k=1}^{\infty} \int_{-\pi}^{\pi} a_k \cos(kx)\,dx + \sum_{k=1}^{\infty} \int_{-\pi}^{\pi} b_k \sin(kx)\,dx. \end{aligned}$$

But for $k \geq 1$, thinking of the integral as an area shows that

$$\int_{-\pi}^{\pi} \sin(kx)\,dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \cos(kx)\,dx = 0,$$

so all terms drop out except the first, giving

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} a_0\,dx = a_0 x \Big|_{-\pi}^{\pi} = 2\pi a_0.$$

Thus, we get the following result:

$$\boxed{a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx.}$$

Thus a_0 is the average value of f on the interval $[-\pi, \pi]$.

To determine the values of any of the other a_k or b_k (for positive k), we use a rather clever method that depends on the following facts. For all integers k and m,

$$\int_{-\pi}^{\pi} \sin(kx)\cos(mx)\,dx = 0,$$

and, provided $k \neq m$,

$$\int_{-\pi}^{\pi} \cos(kx)\cos(mx)\,dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin(kx)\sin(mx)\,dx = 0.$$

(See Problems 26–30 on page 577.) In addition, provided $m \neq 0$, we have

$$\int_{-\pi}^{\pi} \cos^2(mx)\,dx = \pi \quad \text{and} \quad \int_{-\pi}^{\pi} \sin^2(mx)\,dx = \pi.$$

To determine a_k, we multiply the Fourier series by $\cos(mx)$, where m is any positive integer:

$$f(x)\cos(mx) = a_0 \cos(mx) + \sum_{k=1}^{\infty} a_k \cos(kx)\cos(mx) + \sum_{k=1}^{\infty} b_k \sin(kx)\cos(mx).$$

We integrate this between $-\pi$ and π, term by term:

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = \int_{-\pi}^{\pi}\left(a_0\cos(mx) + \sum_{k=1}^{\infty} a_k\cos(kx)\cos(mx) + \sum_{k=1}^{\infty} b_k\sin(kx)\cos(mx)\right)dx$$

$$= a_0\int_{-\pi}^{\pi}\cos(mx)\,dx + \sum_{k=1}^{\infty}\left(a_k\int_{-\pi}^{\pi}\cos(kx)\cos(mx)\,dx\right)$$

$$+\sum_{k=1}^{\infty}\left(b_k\int_{-\pi}^{\pi}\sin(kx)\cos(mx)\,dx\right).$$

Provided $m \neq 0$, we have $\int_{-\pi}^{\pi}\cos(mx)\,dx = 0$. Since the integral $\int_{-\pi}^{\pi}\sin(kx)\cos(mx)\,dx = 0$, all the terms in the second sum are zero. Since $\int_{-\pi}^{\pi}\cos(kx)\cos(mx)\,dx = 0$ provided $k \neq m$, all the terms in the first sum are zero except where $k = m$. Thus the right-hand side reduces to one term:

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = a_m\int_{-\pi}^{\pi}\cos(mx)\cos(mx)\,dx = \pi a_m.$$

This leads, for each value of $m = 1, 2, 3\ldots,$ to the following formula:

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx.$$

To determine b_k, we multiply through by $\sin(mx)$ instead of $\cos(mx)$ and eventually obtain, for each value of $m = 1, 2, 3\ldots,$ the following result:

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx.$$

Exercises and Problems for Section 10.5

Exercises

Which of the series in Exercises 1–4 are Fourier series?

1. $1 + \cos x + \cos^2 x + \cos^3 x + \cos^4 x + \cdots$
2. $\sin x + \sin(x+1) + \sin(x+2) + \cdots$
3. $\dfrac{\cos x}{2} + \sin x - \dfrac{\cos(2x)}{4} - \dfrac{\sin(2x)}{2} + \dfrac{\cos(3x)}{8} + \dfrac{\sin(3x)}{3} - \cdots$
4. $\dfrac{1}{2} - \dfrac{1}{3}\sin x + \dfrac{1}{4}\sin(2x) - \dfrac{1}{5}\sin(3x) + \cdots$
5. Construct the first three Fourier approximations to the square wave function

$$f(x) = \begin{cases} -1 & -\pi \leq x < 0 \\ 1 & 0 \leq x < \pi. \end{cases}$$

Use a calculator or computer to draw the graph of each approximation.

6. Repeat Problem 5 with the function

$$f(x) = \begin{cases} -x & -\pi \leq x < 0 \\ x & 0 \leq x < \pi. \end{cases}$$

7. What fraction of the energy of the function in Problem 6 is contained in the constant term and first three harmonics of its Fourier series?

For Exercises 8–10, find the n^{th} Fourier polynomial for the given functions, assuming them to be periodic with period 2π. Graph the first three approximations with the original function.

8. $f(x) = x^2, \quad -\pi < x \leq \pi.$
9. $h(x) = \begin{cases} 0 & -\pi < x \leq 0 \\ x & 0 < x \leq \pi. \end{cases}$
10. $g(x) = x, \quad -\pi < x \leq \pi.$

Problems

11. Find the constant term of the Fourier series of the triangular wave function defined by $f(x) = |x|$ for $-1 \le x \le 1$ and $f(x+2) = f(x)$ for all x.

12. Using your result from Problem 10, write the Fourier series of $g(x) = x$. Assume that your series converges to $g(x)$ for $-\pi < x < \pi$. Substituting an appropriate value of x into the series, show that

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{2k-1} = \frac{\pi}{4}.$$

13. **(a)** For $-2\pi \le x \le 2\pi$, use a calculator to sketch:
i) $y = \sin x + \frac{1}{3}\sin 3x$
ii) $y = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x$

(b) Each of the functions in part (a) is a Fourier approximation to a function whose graph is a square wave. What term would you add to the right-hand side of the second function in part (a) to get a better approximation to the square wave?

(c) What is the equation of the square wave function? Is this function continuous?

14. **(a)** Find and graph the third Fourier approximation of the square wave $g(x)$ of period 2π:

$$g(x) = \begin{cases} 0 & -\pi \le x < -\pi/2 \\ 1 & -\pi/2 \le x < \pi/2 \\ 0 & \pi/2 \le x < \pi. \end{cases}$$

(b) How does the result of part (a) differ from that of the square wave in Example 1?

15. Suppose we have a periodic function f with period 1 defined by $f(x) = x$ for $0 \le x < 1$. Find the fourth-degree Fourier polynomial for f and graph it on the interval $0 \le x < 1$. [Hint: Remember that since the period is not 2π, you will have to start by doing a substitution. Notice that the terms in the sum are not $\sin(nx)$ and $\cos(nx)$, but instead turn out to be $\sin(2\pi nx)$ and $\cos(2\pi nx)$.]

16. Suppose f has period 2 and $f(x) = x$ for $0 \le x < 2$. Find the fourth-degree Fourier polynomial and graph it on $0 \le x < 2$. [Hint: See Problem 15.]

17. Suppose that a spacecraft near Neptune has measured a quantity A and sent it to earth in the form of a periodic signal $A\cos t$ of amplitude A. On its way to earth, the signal picks up periodic noise, containing only second and higher harmonics. Suppose that the signal $h(t)$ actually received on earth is graphed in Figure 10.33. Determine the signal that the spacecraft originally sent and hence the value A of the measurement.

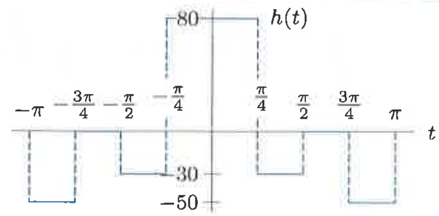


Figure 10.33

18. Figures 10.34 and 10.35 show the waveforms and energy spectra for notes produced by flute and bassoon.[8] Describe the principal differences between the two spectra.

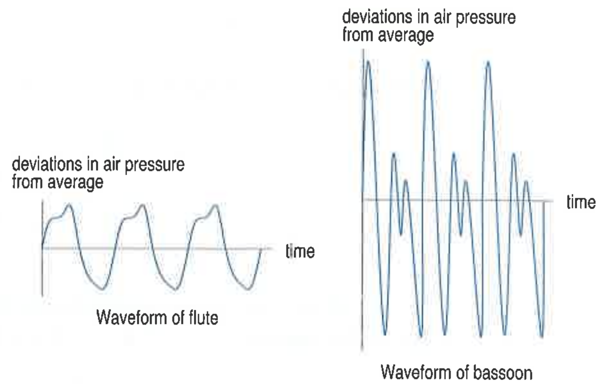


Figure 10.34

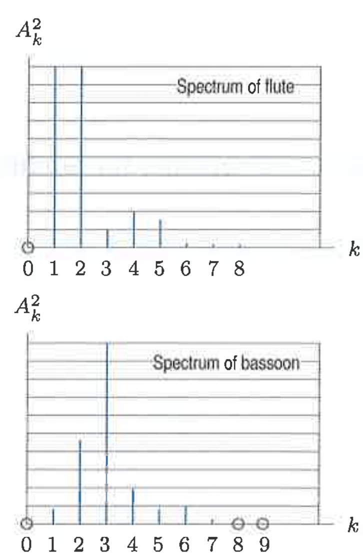


Figure 10.35

19. Show that for positive integers k, the periodic function $f(x) = a_k \cos kx + b_k \sin kx$ of period 2π has energy $a_k^2 + b_k^2$.

[8] Adapted from C.A. Culver, *Musical Acoustics* (New York: McGraw-Hill, 1956), pp. 200, 213.

20. Given the graph of f in Figure 10.36, find the first two Fourier approximations numerically.

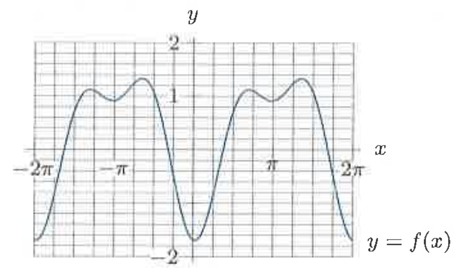


Figure 10.36

21. Justify the formula $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx$ for the Fourier coefficients, b_k, of a periodic function of period 2π. The argument is similar to that in the text for a_k.

In Problems 22–25, the pulse train of width c is the periodic function f of period 2π given by

$$f(x) = \begin{cases} 0 & -\pi \le x < -c/2 \\ 1 & -c/2 \le x < c/2 \\ 0 & c/2 \le x < \pi. \end{cases}$$

22. Suppose that f is the pulse train of width 1.

(a) What fraction of the energy of f is contained in the constant term of its Fourier series? In the constant term and the first harmonic together?
(b) Find a formula for the energy of the k^{th} harmonic of f. Use it to sketch the energy spectrum of f.
(c) How many terms of the Fourier series of f are needed to capture 90% of the energy of f?
(d) Graph f and its fifth Fourier approximation on the interval $[-3\pi, 3\pi]$.

23. Suppose that f is the pulse train of width 0.4.

(a) What fraction of the energy of f is contained in the constant term of its Fourier series? In the constant term and the first harmonic together?
(b) Find a formula for the energy of the k^{th} harmonic of f. Use it to sketch the energy spectrum of f.
(c) What fraction of the energy of f is contained in the constant term and the first five harmonics of f? (The constant term and the first thirteen harmonics are needed to capture 90% of the energy of f.)
(d) Graph f and its fifth Fourier approximation on the interval $[-3\pi, 3\pi]$.

24. Suppose that f is the pulse train of width 2.

(a) What fraction of the energy of f is contained in the constant term of its Fourier series? In the constant term and the first harmonic together?
(b) How many terms of the Fourier series of f are needed to capture 90% of the energy of f?
(c) Graph f and its third Fourier approximation on the interval $[-3\pi, 3\pi]$.

25. After working Problems 22–24, write a paragraph about the approximation of pulse trains by Fourier polynomials. Explain how the energy spectrum of a pulse train of width c changes as c gets closer and closer to 0 and how this affects the number of terms required for an accurate approximation.

For Problems 26–30, use the table of integrals inside the back cover to show that the following statements are true for positive integers k and m.

26. $\displaystyle\int_{-\pi}^{\pi} \cos(kx)\cos(mx)\,dx = 0, \quad \text{if } k \ne m.$

27. $\displaystyle\int_{-\pi}^{\pi} \cos^2(mx)\,dx = \pi.$

28. $\displaystyle\int_{-\pi}^{\pi} \sin^2(mx)\,dx = \pi.$

29. $\displaystyle\int_{-\pi}^{\pi} \sin(kx)\cos(mx)\,dx = 0.$

30. $\displaystyle\int_{-\pi}^{\pi} \sin(kx)\sin(mx)\,dx = 0, \quad \text{if } k \ne m.$

31. Suppose that $f(x)$ is a periodic function with period b. Show that

(a) $g(t) = f(bt/2\pi)$ is periodic with period 2π and $f(x) = g(2\pi x/b)$.
(b) The Fourier series for g is given by

$$g(t) = a_0 + \sum_{k=1}^{\infty}\left(a_k\cos(kt) + b_k\sin(kt)\right)$$

where the coefficients a_0, a_k, b_k are given in the box on page 572.

(c) The Fourier series for f is given by

$$f(x) = a_0 + \sum_{k=1}^{\infty}\left(a_k\cos\left(\frac{2\pi kx}{b}\right) + b_k\sin\left(\frac{2\pi kx}{b}\right)\right)$$

where the coefficients are the same as in part (b).

Strengthen Your Understanding

In Problems 32–33, explain what is wrong with the statement.

32. $\int_{-\pi}^{\pi} \sin(kx)\cos(mx)\,dx = \pi$, where k, m are both positive integers.

33. In the Fourier series for $f(x)$ given by $a_0 + \sum_{k=1}^{\infty} a_k\cos(kx) + \sum_{k=1}^{\infty} b_k\sin(kx)$, we have $a_0 = f(0)$.

In Problems 34–35, give an example of:

34. A function, $f(x)$, with period 2π whose Fourier series has no sine terms.

35. A function, $f(x)$, with period 2π whose Fourier series has no cosine terms.

36. True or false? If f is an even function, then the Fourier series for f on $[-\pi, \pi]$ has only cosines. Explain your answer.

37. The graph in Figure 10.37 is the graph of the first three terms of the Fourier series of which of the following functions?

(a) $f(x) = 3(x/\pi)^3$ on $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$

(b) $f(t) = |x|$ on $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$

(c) $f(x) = \begin{cases} -3 & , \quad -\pi < x < 0 \\ 3 & , \quad 0 < x < \pi \end{cases}\}$ and $f(x + 2\pi) = f(x)$

(d) $f(x) = \begin{cases} \pi + x & , \quad -\pi < x < 0 \\ \pi - x & , \quad 0 < x < \pi \end{cases}\}$ and $f(x + 2\pi) = f(x)$

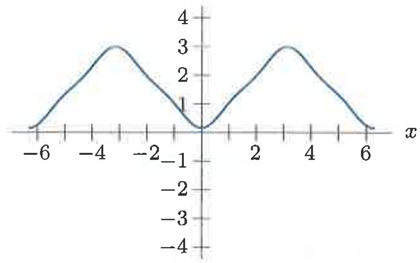


Figure 10.37

CHAPTER SUMMARY (see also Ready Reference at the end of the book)

- **Taylor series and polynomials**
  General expansion about $x = 0$ or $x = a$; specific series for $e^x, \sin x, \cos x, (1 + x)^p$; using known Taylor series to find others by substitution, multiplication, integration, and differentiation; interval of convergence; error in Taylor polynomial expansion
- **Fourier series**
  Formula for coefficients on $[-\pi, \pi]$, $[-b, b]$; Energy theorem

REVIEW EXERCISES AND PROBLEMS FOR CHAPTER TEN

Exercises

For Exercises 1–4, find the second-degree Taylor polynomial about the given point.

1. $e^x, \quad x = 1$ **2.** $\ln x, \quad x = 2$

3. $\sin x, \quad x = -\pi/4$ **4.** $\tan\theta, \quad \theta = \pi/4$

5. Find the third-degree Taylor polynomial for $f(x) = x^3 + 7x^2 - 5x + 1$ at $x = 1$.

For Exercises 6–8, find the Taylor polynomial of degree n for x near the given point a.

6. $\dfrac{1}{1-x}, \quad a = 2, \quad n = 4$

7. $\sqrt{1+x}, \quad a = 1, \quad n = 3$

8. $\ln x, \quad a = 2, \quad n = 4$

9. Write out P_7, the Taylor polynomial of degree $n = 7$ approximating g near $x = 0$, given that

$$g(x) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}3^i}{(i-1)!} x^{2i-1}.$$

10. Find the first four nonzero terms of the Taylor series around $x = 0$ for $f(x) = \cos^2 x$. [Hint: $\cos^2 x = 0.5\,(1 + \cos 2x)$.]

In Exercises 11–18, find the first four nonzero terms of the Taylor series about the origin of the given functions.

11. $t^2 e^t$ **12.** $\cos(3y)$

13. $\theta^2 \cos\theta^2$ **14.** $\sin t^2$

15. $\dfrac{t}{1+t}$ **16.** $\dfrac{1}{1-4z^2}$

17. $\dfrac{1}{\sqrt{4-x}}$ **18.** $\dfrac{z^2}{\sqrt{1-z^2}}$