Growth and asymptotics of functions **MATH 102**
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As we have seen in class, the behaviour of an improper integral $\int_a^\infty f(x)\,dx$ heavily depends upon what the integrand $f(x)$ does as x tends to ∞ (or $-\infty$ for $\int_{-\infty}^b f(x)\,dx$). To aid in our study of this 'long-term' behaviour of functions, we introduce two notations: little-o notation and asymptotic notation.

Note, throughout we assume that all functions take only positive values.

Asymptotic notation

Let f and g be (positive-valued) continuous functions. We say that f and g are *asymptotic* (writing $f \sim g$) if

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = 1.$$

Intuitively, this limit being 1 tells us that in the battle for the product $f(x)\cdot\frac{1}{g(x)}$, neither side wins: as x gets large, the value of $g(x)$ tends to something that perfectly cancels the value that f tends to. For comparison, think about what happens to the following quotients as $x \to +\infty$:

$$\frac{x^2+1}{x}, \quad \frac{2}{e^x}, \quad \frac{2x+1}{3x+19}, \quad \frac{14x+3}{2x+11}.$$

(Respectively, these tend to $+\infty$, 0, $\frac{2}{3}$ and 7: either the numerator or denominator 'wins' and drags the limit to be bigger or smaller than 1). When $f \sim g$, we think of f and g as growing at the same rate as each other.

Little-o notation

Pushing the idea of functions being asymptotic further, we develop 'little-o' notation. Again, let f and g be two positive-valued continuous functions. We say that 'f is *little-o* of g' (writing $f = o(g)$) if

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = 0.$$

In this instance, we see that the $g(x)$ wins the battle in the long run, and outpaces the rate at which f is growing, to drag the value of the quotient down to 0 in the limit. Thus, if $f = o(g)$, we think of g as having a much faster rate of growth than f.

Examples of growth rates

All of our favorite functions fit into a hierarchy of growth. Let c be a constant function, $\log_b$ be the logarithm function base b, $a(x) = x^r$ for some real $r > 0$, and $\exp_b$ be the exponential function base b. Then, we can check, that:

- $c = o(\log_b)$,

- $\log_b = o(a)$,
- $a = o(\exp_b)$.

The moral here is that of these four types of functions, as $x \to +\infty$, constant functions grow the slowest, being out-paced by logarithms, which are out-paced by powers of x, which are out-paced by exponentials.

The algebra of little-o

The power of little-o and asymptotic notation arises when we combine the two. For instance, the following asymptotic identities hold:

1. If $f = o(g)$ and b and c are constants, then
$$\boxed{b \cdot g + c \cdot f \sim b \cdot g}$$
(i.e. g grows so much more quickly than f that adding $c \cdot f$ has no effect on the asymptotic value of $b \cdot g$).
2. If $f_1 \sim g_1$ and $f_2 \sim g_2$, then
$$\boxed{\frac{f_1}{f_2} \sim \frac{g_1}{g_2}}$$
and
$$\boxed{f_1 \cdot g_1 \sim f_2 \cdot g_2}.$$
3. If $f \sim g$ and n a fixed real number, then
$$\boxed{f^n \sim g^n}.$$
4. For a constant c, if $f = o(g)$, then
$$\boxed{f = o(c \cdot g)}$$
and
$$\boxed{c \cdot f = o(g)}.$$
5. If $g_1 \sim g_2$ and $f = o(g_1)$, then
$$\boxed{f = o(g_2)}.$$

These rules allow us to determine the asymptotic behavior of complicated looking functions by breaking them up into pieces that we understand better. For example, consider the quotient

$$f(x) = \frac{\sqrt{6x^7 - 5x - 1} + \ln(x)}{x^5 + \sin(2x)}.$$

We will find easier to analyze functions, one asymptotic with the numerator of $f(x)$ and one asymptotic with the denominator, and use Rule (2) to replace the quotient $f(x)$ with a simpler one.

First, the numerator. Since the leading term of $6x^7 - 5x - 1$ dominates as x grows, we have that

$$6x^7 - 5x - 1 \sim 6x^7.$$

Applying Rule (3) with $n = \frac{1}{2}$, we get

$$\sqrt{6x^7 - 5x - 1} \sim \sqrt{6x^7} = \sqrt{6}x^{\frac{7}{2}}.$$

Rule (5) thus gives us that $\ln(x)$ is little-o of both $\sqrt{6x^7 - 5x - 1}$ and $\sqrt{6}x^{\frac{7}{2}}$. Applying Rule (1) gives that the numerator of $f(x)$ is asymptotic with $\sqrt{6}x^{\frac{7}{2}}$.

Similarly, since $\sin(2x)$ is bounded in value, $\sin(2x) = o(x^5)$, and so the denominator is asymptotic with x^5 by Rule (1). Finally, by Rule (2), we see that $f(x)$ is asymptotic with the simpler quotient

$$\frac{\sqrt{6}x^{\frac{7}{2}}}{x^5} = \frac{\sqrt{6}}{x^{\frac{3}{2}}}.$$

Application to improper integrals

Like the comparison integrals we saw in class, we can use asymptotic information to determine if an improper integral converges or diverges, as follows.

If there exists a constant c such that $f \sim c \cdot g$, then the improper integrals of f and g on some interval $[a, +\infty]$ either both converge or both diverge.

For instance, since our example $f(x)$ is the previous section is asymptotic with $\sqrt{6}x^{-\frac{3}{2}}$, once we check that the improper integral of $\sqrt{6}x^{-\frac{3}{2}}$ on $[a, +\infty]$ (for $a > 0$) converges (easy!), we know that the improper integral of $f(x)$ also converges.

The selling point!

This method using little-o and $\sim$ has one major advantage over the method using comparison integrals that we saw in class: you use it by directly simplifying the integrand $f(x)$ in your improper integrand (writing $\sim$ instead of $=$, of course!). To use comparison integrals, you need to explore somewhat blindly for an integrand strictly smaller (or larger) than $f(x)$, that you can integrate more easily. As an example, we saw above using little-o that the improper integral of the quotient $f(x)$ given on the previous page converges — can you easily find a comparison integral that tells us this same result?

On the other hand, sometimes we need to resort to a comparison integral. For example, for

$$g(x) = \frac{\sin^2(x)}{x^5},$$

the numerator is not asymptotic to anything helpful (since $\sin^2$ oscillates between 0 and 1 forever), so to compute an improper integral of $g(x)$, we need to use a comparison integral.