wrote on that occasion. The second author would like to thank the University of Warwick; he was a visitor there when the final version of the book was completed and he used the University's computer facilities extensively. We also acknowledge some valuable suggestions concerning chapter E made by C. C. Adams. Lastly, we warmly thank Andrea Petronio for the very accurate illustrations and David Trotman for his help in checking our English.

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Pisa, May 1992

Chapter A. Hyperbolic Space

This chapter is devoted to the definition of a Riemannian n-manifold $\mathbb{H}^n$ called hyperbolic n-space and to the determination of its geometric properties (isometries, geodesics, curvature, *etc.*). This space is the local model for the class of manifolds we shall deal with in the whole book. The results we are going to prove may be found in several texts (*e.g.* [Bea], [Co], [Ep2], [Fe], [Fo], [Greenb2], [Mag], [Mask2], [Th1, ch. 3] and [Wol]) so we shall omit precise references. The line of the present chapter is partially inspired by [Ep2], though we shall be dealing with a less general situation. For a wide list of references about hyperbolic geometry from ancient times to 1980 we address the reader to [Mi3].

A.1 Models for Hyperbolic Space

Let n be a fixed natural number. In order to avoid trivialities we shall always assume $n \geq 2$. We shall give different models for a real Riemannian n-manifold denoted by $\mathbb{H}^n$, which we shall call hyperbolic n-space; these models will be by construction isometrically diffeomorphic to each other. We shall introduce different symbols for them, and we shall use these symbols in order to emphasize a concrete representation of the manifold, while the symbol $\mathbb{H}^n$ will be used for the abstract manifold. We shall not get involved in categorial definitions: every Riemannian manifold isometrically diffeomorphic to $\mathbb{H}^n$ will be identified with $\mathbb{H}^n$.

HYPERBOLOID MODEL. In $\mathbb{R}^{n+1}$ let us consider the standard symmetric bi-linear form of signature $(n,1)$:

$$\langle x|y\rangle_{(n,1)} = \sum_{i=1}^{n} x_i \cdot y_i - x_{n+1} \cdot y_{n+1}$$

and let us consider the upper fold of the hyperboloid naturally associated to $\langle\cdot|\cdot\rangle_{(n,1)}$:

$$\mathrm{I}_n = \left\{x \in \mathbb{R}^{n+1} : \langle x|x\rangle_{(n,1)} = -1,\ x_{n+1} > 0\right\}.$$

Since I_n is the pre-image of a regular value of a differentiable function, it is a differentiable oriented hypersurface in $\mathbb{R}^{n+1}$; in particular it is endowed with a differentiable structure of dimension n. For $x \in \mathrm{I}_n$ the tangent space to I_n in x is given by

$$T_x \mathrm{I}_n = \{y \in \mathbb{R}^{n+1} : \langle x|y\rangle_{(n,1)} = 0\} = \{x\}^{\perp}.$$

Since $\langle x|x\rangle_{(n,1)} = -1$, the restriction of $\langle .|.\rangle_{(n,1)}$ to $\{x\}^{\perp}$ is positive-definite, *i.e.* it is a scalar product on $\{x\}^{\perp}$. So, a metric is naturally defined on the tangent space to each point of I_n; it is easily verified that this metric is globally differentiable, and therefore I_n is endowed with a Riemannian structure. We shall denote by $\mathbb{I}^n$ the manifold I_n endowed with this structure.

Disc Model. Let π be the restriction to $\mathbb{I}^n$ of the stereographic projection with respect to $(0, ..., 0, -1)$ of $\{x \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$ onto $\mathbb{R}^n \times \{0\}$. We omit the last coordinate, so that the range of π is $\mathbb{R}^n$:

$$\pi(x) = \frac{(x_1, ..., x_n)}{1 + x_{n+1}}.$$

It is easily verified that π is a diffeomorphism of $\mathbb{I}^n$ onto the open Euclidean unit ball D^n of $\mathbb{R}^n$. The manifold D^n endowed with the pull-back metric with respect to π^{-1} will be denoted by $\mathbb{D}^n$. This manifold is canonically oriented as a domain of $\mathbb{R}^n$.

Half-space Model. Let us consider the differentiable mapping:

$$i : \mathbb{D}^n \to \mathbb{R}^n \qquad x \mapsto 2\frac{x + e_n}{\|x + e_n\|^2} - e_n$$

where $e_n = (0, ..., 0, 1)$ and $\|.\|$ denotes the Euclidean norm on $\mathbb{R}^n$. (In Sect. A.3 we shall introduce the notion of inversion with respect to a sphere: it is worth remarking early that i is the inversion with respect to the sphere of centre $-e_n$ and radius $\sqrt{2}$.) It is easily checked that i is a diffeomorphism of $\mathbb{D}^n$ onto the open half-space $\Pi^{n,+} = \{x \in \mathbb{R}^n : x_n > 0\}$. We shall denote by $\mathbb{H}^{n,+}$ this half-space endowed with the pull-back metric with respect to i^{-1}. $\mathbb{H}^{n,+}$ is canonically oriented as a domain of $\mathbb{R}^n$.

Projective (or Klein) Model. Let p be the restriction to $\mathbb{I}^n$ of the canonical projection of $\mathbb{R}^{n+1}$ onto the real projective n-space $\mathbb{R}\mathbb{P}^n$. p is a diffeomorphism onto an open subset of $\mathbb{R}\mathbb{P}^n$ (actually, the unit disc in a suitable affine chart of $\mathbb{R}\mathbb{P}^n$) which can be endowed with the pull-back metric with respect to p^{-1}. Since we are not going to use this model we do not introduce a specific symbol for this representation of $\mathbb{H}^n$.

Figures 1 and 2 illustrate the geometric construction of the first three models in the 2-dimensional case.

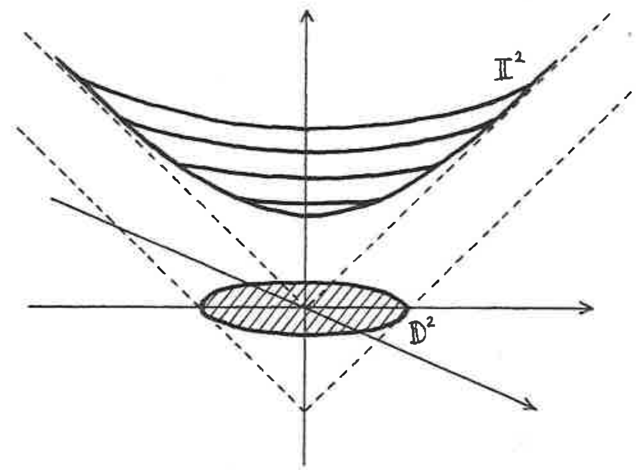

Fig. A.1. Two-dimensional models of hyperbolic space: the hyperboloid and its projection onto the disc

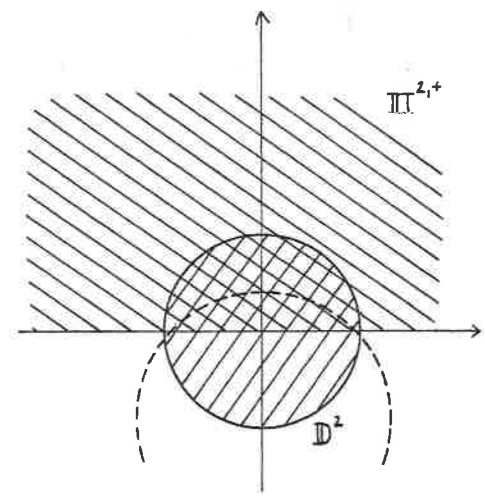

Fig. A.2. Two-dimensional models of hyperbolic space: the disc and its inversion onto the upper half-plane

A.2 Isometries of Hyperbolic Space: Hyperboloid Model

For a Riemannian manifold M we shall denote by $\mathcal{I}(M)$ the set of all isometric diffeomorphisms of M onto itself (briefly: isometries of M). If M is supposed to be oriented, we shall denote by $\mathcal{I}^+(M)$ the set of all isometries of M preserving orientation. $\mathcal{I}(M)$ and $\mathcal{I}^+(M)$ are groups with respect to the operation of composition. In this section we shall determine the groups $\mathcal{I}(\mathbb{I}^n)$ and $\mathcal{I}^+(\mathbb{I}^n)$, while the isometries of $\mathbb{H}^n$ in the other models will be calculated later.

We shall denote the differential of a mapping f in a point x of M by $d_x f$. The scalar product defined on the tangent space $T_x M$ will be denoted by $\langle .|. \rangle_x$ and the quadratic form associated to it by ds_x^2. We recall that the condition that f be an isometry means the following:

$$\langle d_x f(v) | d_x f(w) \rangle_{f(x)} = \langle v|w \rangle_x \qquad \forall x \in M,\ v, w \in T_x M.$$

The following result is quite standard, but it will be included for completeness since it is the basis for most of our arguments; we shall use the notion of geodesic and exponential mapping, and the well-known result about existence of normal neighborhoods (see *e.g.* [He]).

Proposition A.2.1. Let M and N be Riemannian manifolds of the same dimension, assume M is connected and let

$$\phi_1 : M \to \phi_1(M) \subseteq N, \quad \phi_2 : M \to \phi_2(M) \subseteq N$$

be local isometries onto their range. If for some $y \in M$ we have $\phi_1(y) = \phi_2(y)$ and $d_y\phi_1 = d_y\phi_2$ then $\phi_1 = \phi_2$.
The conclusion holds in particular if ϕ_1 and ϕ_2 are isometries of M onto N.

Proof. The set

$$S = \{x \in M : \phi_1(x) = \phi_2(x), d_x\phi_1 = d_x\phi_2\}$$

is obviously closed and it contains y, hence we only have to prove that it is open. Let $x \in S$; since M and N have the same dimension the ranges of ϕ_1 and ϕ_2 are open in N; it follows that we can find an open neighborhood V of $\phi_1(x) = \phi_2(x)$ and two open neighborhoods U_1, U_2 of x such that $\phi_i : U_i \to V$ is a surjective isometry for $i = 1, 2$. Let $U \subseteq U_1$ be a normal neighborhood of x and $p : T_x M \supset W \to U$ be the corresponding restriction of the exponential mapping. We set $f = \left(\phi_2|_{U_2}\right)^{-1} \circ \left(\phi_1|_{U_1}\right)$; f is an isometry of U_1 onto U_2, $f(x) = x$ and $d_x f = \mathrm{I}$. If γ is a geodesic arc in U starting at x then $f \circ \gamma$ is a geodesic arc starting at x with tangent vector

$$(f \circ \gamma)'(0) = d_x f(\gamma'(0)) = \gamma'(0)$$

and hence $f \circ \gamma = \gamma$, which implies $f \circ p = p$, and finally $f|_U = \mathrm{id}$. It was checked that $\phi_1|_U = \phi_2|_U$, so that $U \subset S$, and the proposition is proved. □

In order to illustrate completely the determination of the isometries of $\mathbb{I}^n$ we start with some elementary facts in linear algebra.

Let V be an n-dimensional real vector space and let $\langle .|. \rangle$ be a non-degenerate bi-linear form on V. It is well-known that there exists a basis $\{v_1, ..., v_n\}$ of V such that

$$\langle v_i | v_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ +1 & \text{if } i = j \leq p \\ -1 & \text{if } i = j > p\,. \end{cases}$$

If we set $q = n - p$ the pair (p, q) depends only on $\langle .|. \rangle$ and it is called the signature. If $\mathrm{Gl}(V)$ denotes the linear group on V we set

$$\mathrm{O}(V, \langle .|. \rangle) = \{A \in \mathrm{Gl}(V) : \langle Ax | Ay \rangle = \langle x|y \rangle\ \forall x, y \in V\}.$$

If $V = W \oplus W'$ and $p : V \to W$ is the associated projection, we shall call reflection the linear mapping $\rho : v \mapsto 2p(v) - v$. It is easily verified that ρ is in $\mathrm{O}(V, \langle .|. \rangle)$ if and only if $W' = W^\perp$, where $\perp$ denotes orthogonality with respect to $\langle .|. \rangle$. If this is the case we shall say that ρ is the reflection with respect to W, or parallel to $W^\perp$. From now on we shall call reflections only those with respect to a hyperplane (*i.e.* parallel to a vector v such that $\langle v|v \rangle \neq 0$).

Proposition A.2.2. $\mathrm{O}(V, \langle .|. \rangle)$ is generated by reflections.

Proof. Let us remark first that $-\mathrm{I}$ is generated by the n reflections parallel to the vectors v_i of the basis described above.

We carry out the proof by induction on the dimension n. The first step is obvious. Assume the proposition is true for an integer n, let V have dimension $n + 1$, let $\langle .|. \rangle$ be a non-degenerate bi-linear form on V, let A belong to $\mathrm{O}(V, \langle .|. \rangle)$ and choose $v \in V$ such that $\langle v|v \rangle \neq 0$.
We can assume $\langle Av - v | Av - v \rangle \neq 0$: if this is not the case it is easily verified that $\langle -Av - v | -Av - v \rangle \neq 0$, hence we can replace A by $-A$; but by the first remark if $-A$ is a product of reflections then A is too. Let ρ be the reflection parallel to $Av - v$; since

$$v = \frac{1}{2}(Av + v) - \frac{1}{2}(Av - v) \qquad \langle Av + v | Av - v \rangle = 0$$

then $\rho(v) = Av \Rightarrow (\rho \circ A)(v) = v \Rightarrow (\rho \circ A)\big|_{v^\perp} \in \mathrm{O}(v^\perp, \langle .|. \rangle\big|_{v^\perp \times v^\perp})$. Since every reflection in $v^\perp$ extends to a reflection in V the induction hypothesis implies that A is a product of reflections. □

Assume now that $V = \mathbb{R}^{n+1}$ and $\langle .|. \rangle$ is the standard bi-linear form of signature $(n, 1)$, and let I_n be defined as in Sect. A.1. We shall denote by $\mathrm{O}(\mathrm{I}_n)$ the subgroup of $\mathrm{O}(\mathbb{R}^{n+1}, \langle .|. \rangle)$ of those mappings that keep I_n invariant, and by $\mathrm{SO}(\mathrm{I}_n)$ the intersection of $\mathrm{O}(\mathrm{I}_n)$ with $\mathrm{Sl}(n + 1, \mathbb{R})$. Let us remark that $\mathrm{O}(\mathrm{I}_n)$ and $\mathrm{SO}(\mathrm{I}_n)$ are closed subgroups of $\mathrm{Gl}(n + 1, \mathbb{R})$, and hence they are naturally endowed with a Lie group structure.

Proposition A.2.3. $\mathrm{O}(\mathrm{I}_n)$ is generated by the reflections it contains.

Proof. Every reflection parallel to a vector v with $\langle v|v \rangle \neq 0$ keeps the whole hyperboloid $\mathrm{I}_n \cup (-\mathrm{I}_n)$ invariant, and it exchanges the two folds if and only if $\langle v|v \rangle < 0$.

Let $A \in \mathrm{O}(\mathrm{I}_n)$, and write it as a product of reflections: $A = \rho_1 \circ ... \circ \rho_k$. Let ρ_i be parallel to a vector x_i; if $\langle x_i | x_i \rangle < 0$ we can complete x_i to an orthogonal basis $\{x_i, w_1, \ldots, w_n\}$ of $\mathbb{R}^{n+1}$, with the property that $\langle w_j|w_j\rangle > 0 \,\forall j$. Then ρ_i is given by $-(\sigma_1 \circ \ldots \circ \sigma_n)$, where σ_j is the reflection parallel to w_j. If we make this substitution for all i's such that $\langle x_i|x_i\rangle < 0$ we obtain that

$$A = \pm(\tau_1 \circ \ldots \circ \tau_h)$$

where all the τ_k's are reflections and belong to $O(I_n)$. The minus sign is obviously absurd and hence the proposition is proved. □

Theorem A.2.4. $\mathcal{I}(\mathbb{I}^n)$ consists of the restrictions to $\mathbb{I}^n$ of the elements of $O(I_n)$, whence $\mathcal{I}(\mathbb{I}^n) \cong O(I_n)$; in particular $\mathcal{I}(\mathbb{I}^n)$ is generated by reflections. Similarly $\mathcal{I}^+(\mathbb{I}^n) \cong SO(I_n)$.

Proof. Let $f \in \mathcal{I}(\mathbb{I}^n)$ and choose arbitrarily $x \in \mathbb{I}^n$; since $d_x f$ is an isometry of $x^\perp$ onto $f(x)^\perp$ and $\langle x|x\rangle = \langle f(x)|f(x)\rangle = -1$ it is readily checked that the linear mapping

$$A : \mathbb{R}^{n+1} = \mathbb{R}x \oplus x^\perp \to \mathbb{R}^{n+1} \qquad \lambda x + v \mapsto \lambda f(x) + d_x f(v)$$

is an element of $O(I_n)$. As the restriction of A to $\mathbb{I}^n$ is obviously an isometry, and $f(x) = Ax$, $d_x f = A|_{T_x\mathbb{I}^n}$, then by Proposition A.2.1 f is the restriction of A to $\mathbb{I}^n$. It follows that

$$\mathcal{I}(\mathbb{I}^n) = \{A|_{\mathbb{I}^n} : A \in O(I_n)\}.$$

Since the linear span of I_n is $\mathbb{R}^{n+1}$, the mapping

$$O(I_n) \ni A \mapsto A|_{I_n} \in \mathcal{I}(\mathbb{I}^n)$$

is one-to-one, and hence it is a group isomorphism.

The case of orientation-preserving isometries is a straight-forward consequence of the general one. □

Though we are mainly interested in hyperbolic space we prove an analogue of Theorem A.2.4 for two other very important Riemannian manifolds: the sphere and Euclidean space.

$\mathbb{R}^n$ will be endowed with the standard Euclidean metric, and the unit sphere S^n in $\mathbb{R}^{n+1}$ will be endowed with the restriction of the Euclidean metric to its tangent bundle (the construction is completely analogous to the one we presented in A.1 for $\mathbb{I}^n$: $\langle\cdot|\cdot\rangle_{(n,1)}$ is substituted by the Euclidean metric and -1 by 1). In $\mathbb{R}^n$ reflections with respect to affine hyperplanes are naturally defined, while in S^n we shall consider the restrictions of the reflections of $\mathbb{R}^{n+1}$.

Theorem A.2.5. $\mathcal{I}(S^n) = \{A|_{S^n} : A \in O(n+1)\}$,
$\mathcal{I}(\mathbb{R}^n) = \{(x \mapsto Ax + b : A \in O(n), b \in \mathbb{R}^n\}$.
Both of these groups are generated by reflections.

Proof. The technique is the same as for A.2.4: inclusions $\supseteq$ are obvious, and for the converse it is checked that for each element of the group on the left

an element of the group on the right can be found in such a way that the two coincide up to first order.

The last assertion is obvious in the first case, while in the second we only have to remark that the translation of a vector b is the product of the reflections with respect to $b^\perp$ and $b/2 + b^\perp$. □

A.3 Conformal Geometry

In this section we will be concerned with conformal geometry in $\mathbb{R}^n$ and we shall prove an important theorem due to Liouville (see for instance [Ber]). The reason for this long parenthesis is that conformal geometry in $\mathbb{R}^n$ permits a complete calculation of the isometries of $\mathbb{H}^n$ in the disc and half-space model: we shall prove that every isometry with respect to the hyperbolic structure in $\mathbb{D}^n$ and $\Pi^{n,+}$ is a conformal automorphism with respect to the Euclidean structure naturally defined by the immersion in $\mathbb{R}^n$, and conversely.

Let M and N be Riemannian manifolds: we shall say a diffeomorphism $f : M \to N$ is conformal if there exists a differentiable positive function α on M such that

$$\langle d_x f(v)|d_x f(w)\rangle_{f(x)} = \alpha(x)\langle v|w\rangle_x \quad \forall x \in M,\ v, w \in T_x M$$

(*i.e.* f preserves angles but not necessarily lengths). This definition can be easily generalized to manifolds endowed with a conformal structure, *i.e.* manifolds in which the angle between two vectors is defined.

The set of conformal diffeomorphisms of M onto N will be denoted by $\mathrm{Conf}(M,N)$, and by $\mathrm{Conf}(M)$ in case $N = M$; remark that $\mathrm{Conf}(M)$ is a group. As usual, the + superscript will mean that orientation (if any) is preserved.

We introduce now a very important notion for the study of conformal geometry in $\mathbb{R}^n$. If $x_0 \in \mathbb{R}^n$ and $\alpha > 0$ we shall call inversion with respect to the sphere $M(x_0, \alpha)$ of centre x_0 and radius $\sqrt{\alpha}$ the following mapping:

$$i_{x_0,\alpha} : x \mapsto \alpha \cdot \frac{x - x_0}{\|x - x_0\|^2} + x_0.$$

We shall think of $i_{x_0,\alpha}$ both as a mapping of $\mathbb{R}^n \setminus \{x_0\}$ onto itself and as a mapping of $\mathbb{R}^n \cup \{\infty\}$ onto iself, where $\mathbb{R}^n \cup \{\infty\} \cong S^n$ is the one-point compactification of $\mathbb{R}^n$, and $i_{x_0,\alpha}$ exchanges x_0 and ∞. Throughout this section S^n will be endowed with its natural conformal structure; remark that $\mathbb{R}^n = S^n \setminus \{\infty\}$ inherits from S^n its own conformal structure; every open subset of $\mathbb{R}^n$ will be endowed with the conformal structure induced from $\mathbb{R}^n$. Remark that the definition of $i_{x_0,\alpha}$ makes sense also for $\alpha < 0$, and it is easily checked that in this case $i_{x_0,\alpha}$ is the composition of the inversion with respect to $M(x_0, -\alpha)$ with the symmetry centred at x_0. In the following proposition we shall list a few important properties of inversions. We shall say two hyperplanes H_1 and H_2 in $\mathbb{R}^n$ are orthogonal if the lines $H_1^\perp$ and $H_2^\perp$ are orthogonal; consequently we shall say that two intersecting spheres are orthogonal if for any point of their intersection the two tangent hyperplanes are orthogonal in the above sense; that is, if x_0 and x_1 are the centres of the spheres, for each point x of the intersection $\langle x - x_0 | x - x_1 \rangle = 0$.

We are not going to say explicitly if an inversion $i_{x_0,\alpha}$ is considered to be defined on $\mathbb{R}^n \setminus \{x_0\}$ or on S^n, since it will be evident from the context.

Proposition A.3.1. (1) $i_{x_0,\alpha} \circ i_{x_0,\beta}$ is the dilation centred at x_0 of ratio α/β.
(2) $i_{x_0,\alpha}$ is a C^∞ involution (of both $\mathbb{R}^n \setminus \{x_0\}$ and S^n).
(3) $i_{x_0,\alpha}|_{M(x_0,\alpha)} = \mathrm{id}$.
(4) $i_{x_0,\alpha}$ is a conformal mapping.
(5) Given $\alpha, \beta > 0$ and $x_1 \neq x_0$ the following facts are equivalent:
  i) $M(x_1,\beta)$ is $i_{x_0,\alpha}$-invariant;
  ii) $M(x_0,\alpha)$ is $i_{x_1,\beta}$-invariant;
  iii) $\|x_1 - x_0\|^2 = \alpha + \beta$;
  iv) $M(x_1,\beta)$ and $M(x_0,\alpha)$ are orthogonal spheres.
(6) Let $i = i_{x_0,\alpha}$; then
  i) H hyperplane, $H \ni x_0 \Rightarrow i(H) = H$;
  ii) H hyperplane, $H \not\ni x_0 \Rightarrow i(H)$ sphere, $i(H) \ni x_0$;
  iii) M sphere, $M \ni x_0 \Rightarrow i(M)$ hyperplane, $i(M) \not\ni x_0$,
  iv) M sphere, $M \not\ni x_0 \Rightarrow i(M)$ sphere, $i(M) \not\ni x_0$;
  v) i operates bijectively on the set of all open balls and all open half-spaces in $\mathbb{R}^n$.

Proof. First of all we remark that if T is the translation $x \mapsto x + x_0$ we have $i_{x_0,\alpha} = T \circ i_{0,\alpha} \circ T^{-1}$, and $i_{0,\alpha} = \alpha \cdot i_{0,1}$, hence we shall often assume $x_0 = 0$ and $\alpha = 1$.

(1) $$(i_{0,\alpha} \circ i_{0,\beta})(x) = \alpha \frac{\beta x / \|x\|^2}{\left\| \beta x / \|x\|^2 \right\|^2} = \alpha\beta^{-1} x.$$

(2) By (1) $i_{x_0,\alpha}$ is an involution; differentiability is evident.

(3) Obvious.

(4) Dilations and translations are conformal, and hence we refer to $i_{0,1}$, which is conformal at $x \neq 0$ since its differential is
$$d_x i_{0,1}(y) = \frac{1}{\|x\|^2} \cdot P_x(y)$$
where P_x is the reflection parallel to x, *i.e.* the reflection with respect to the hyperplane $x^\perp$. Moreover $i_{0,1}$ is the standard chart around ∞, and hence it is by definition conformal at 0.

(5)

i) $\Rightarrow$ iii). We assume $x_0 = 0$ and $\alpha = 1$. The intersection of $M(x_1,\beta)$ with the line $\mathbb{R} x_1$ consists of the points $(1 \pm \sqrt{\beta}/\|x_1\|)x_1$ and hence $i_{0,1}$ must

exchange them (in fact both the sphere and the line are $i_{0,1}$-invariant, and it is easily checked that it is impossible that both points are fixed). By direct calculation
$$(1 - \sqrt{\beta}/\|x_1\|)x_1 = i_{0,1}((1 + \sqrt{\beta}/\|x_1\|)x_1) = \frac{x_1}{(1 + \sqrt{\beta}/\|x_1\|)\|x_1\|^2}$$
$$\Rightarrow (1 - \sqrt{\beta}/\|x_1\|) \cdot (1 + \sqrt{\beta}/\|x_1\|) \cdot \|x_1\|^2 = 1 \Rightarrow \|x_1\|^2 = 1 + \beta.$$

iii) $\Rightarrow$ i). As above, $x_0 = 0$ and $\alpha = 1$. Let $x \in M(x_1,\beta)$, then
$$\|x_1\|^2 - 1 = \beta = \|x - x_1\|^2 = \|x\|^2 - 2\langle x|x_1\rangle + \|x_1\|^2 \Rightarrow 2\langle x|x_1\rangle = 1 + \|x\|^2$$
and therefore
$$\|i_{0,1}(x) - x_1\|^2 = \left\| \frac{x}{\|x\|^2} - x_1 \right\|^2 = \frac{1}{\|x\|^2} - \frac{2\langle x|x_1\rangle}{\|x\|^2} + 1 + \beta = \beta.$$

ii) $\Leftrightarrow$ iii) is proved in the very same way.

iii) $\Leftrightarrow$ iv). Since $\alpha + \beta < (\sqrt{\alpha} + \sqrt{\beta})^2$, condition iii) implies that the two spheres intersect. Moreover, if x is in the intersection we have
$$\|x_1 - x_0\|^2 = \alpha + \beta \Leftrightarrow \|x_1 - x\|^2 + \|x_0 - x\|^2 = \|x_1 - x_0\|^2 \Leftrightarrow$$
$$\Leftrightarrow -\langle x_1|x\rangle + \langle x|x\rangle - \langle x_0|x\rangle = -\langle x_1|x_0\rangle \Leftrightarrow \langle x_1 - x | x_0 - x\rangle = 0.$$

(6) Since the properties we are considering are invariant under dilations and translations, we take $i = i_{0,1}$.

i) is obvious.

ii). Let $H = h + h^\perp$ with $h \in \mathbb{R}^n \setminus \{0\}$. We set $c = h/2\|h\|^2, \gamma = 1/4\|h\|^2$. For $x \neq 0$ we have
$$i(x) \in M(c,\gamma) \Leftrightarrow \|i(x) - c\|^2 = \gamma \Leftrightarrow \left\| \frac{x}{\|x\|^2} - \frac{h}{2\|h\|^2} \right\| = \frac{1}{4\|h\|^2} \Leftrightarrow$$
$$\Leftrightarrow \frac{1}{\|x\|^2} - \frac{\langle x|h\rangle}{\|x\|^2 \cdot \|h\|^2} = 0 \Leftrightarrow \langle h - x|h\rangle = 0 \Leftrightarrow x \in H.$$

Moreover $i(\infty) = 0 \in M(c,\gamma)$, whence $i(H) = M(c,\gamma)$.

iii). Let $M = M(c,\gamma)$. Since $0 \in M$ we have $\gamma = \|c\|^2$. If we set $h = c/2\|c\|^2$ and $H = h + h^\perp$, by ii) we have $i(H) = M$, and then $i(M) = H$.

iv). Let $M = M(c,\gamma)$. Since $0 \notin M$ we have $\|c\|^2 \neq \gamma$. The following holds:

$$\begin{aligned}
i(x)\in M(c,\gamma) &\Leftrightarrow \left\|\frac{x}{\|x\|^2}-c\right\|^2=\gamma \Leftrightarrow\\
&\Leftrightarrow \frac{1}{\|x\|^2}-\frac{2\langle x|c\rangle}{\|x\|^2}+\|c\|^2=\gamma \Leftrightarrow\\
&\Leftrightarrow \|x\|^2-\frac{2\langle x|c\rangle}{\|c\|^2-\gamma}+\frac{1}{\|c\|^2-\gamma}=0 \Leftrightarrow\\
&\Leftrightarrow \left\|x-\frac{c}{\|c\|^2-\gamma}\right\|^2=\frac{\|c\|^2}{(\|c\|^2-\gamma)^2}-\frac{1}{\|c\|^2-\gamma}=\frac{\gamma}{(\|c\|^2-\gamma)^2} \Leftrightarrow\\
&\Leftrightarrow x\in M\left({}^{c}/_{(\|c\|^2-\gamma)},{}^{\gamma}/_{(\|c\|^2-\gamma)^2}\right).
\end{aligned}$$

Therefore $i(M(c,\gamma)) = i^{-1}(M(c,\gamma)) = M\left({}^{c}/_{(\|c\|^2-\gamma)},{}^{\gamma}/_{(\|c\|^2-\gamma)^2}\right)$, and by iii) this sphere cannot contain 0.

v). If A is either an open ball or an open half-space we have that ∂A is either a sphere or a hyperplane, and by i)–iv) the same holds for $i(\partial A)$. By (2) $i(A)$ is connected and its boundary is $i(\partial A)$, which implies that it is either a ball or a half-space. □

Now, for $n \geq 2$ we will deal with the set of all conformal diffeomorphisms between two domains of $\mathbb{R}^n$. The technique is completely different for the case $n = 2$ and the case $n \geq 3$; however, for the particular open sets we are interested in, the result is the same for all integers n.

First case: $n = 2$.

We begin by recalling (see [Sp] or [DC]) that a connected oriented Riemannian surface M admits a complex structure (given by isothermal coordinates), and this structure is uniquely determined by the requirement that

$$f:\mathbb{C}\supset U\to M$$

is a holomorphic chart if and only if it preserves orientation and

$$ds^2_{f(z)}(d_zf(w))=\alpha(z)\cdot|w|^2 \qquad \forall\, z\in U, w\in\mathbb{C}$$

for some function $\alpha > 0$.

By the following proposition conformal geometry in dimension 2 reduces to a problem in the theory of functions of one complex variable.

Proposition A.3.2. If M and N are connected oriented Riemannian surfaces (naturally endowed with complex structures), the set of all conformal diffeomorphisms of M onto N is the set of all holomorphisms and all anti-holomorphisms of M onto N.

Proof. This fact could be easily deduced from the uniqueness of the complex structure. However, we shall prove it directly: actually, this very argument proves the uniqueness of the complex structure (while existence is much more complicated).

Let $f : M \to N$ be conformal. Since the only both holomorphic and anti-holomorphic functions are the constants, and since holomorphy and anti-holomorphy are closed conditions, by connectedness it suffices to prove that f is locally holomorphic or anti-holomorphic, hence we can assume that M and N are domains of $\mathbb{C}$. The conformality condition is expressed by

$$\begin{aligned}
&|df|^2=\alpha\cdot|dz|^2\\
\Rightarrow\ &\left|\frac{\partial f}{\partial z}dz+\frac{\partial f}{\partial \bar z}d\bar z\right|^2=\alpha|dz|^2\\
\Rightarrow\ &\left(\left|\frac{\partial f}{\partial z}\right|^2+\left|\frac{\partial f}{\partial \bar z}\right|^2\right)|dz|^2+2\Re\left[\frac{\partial f}{\partial z}\cdot\overline{\left(\frac{\partial f}{\partial \bar z}\right)}\cdot(dz)^2\right]=\alpha|dz|^2.
\end{aligned}$$

Since $2\Re\left[{}^{\partial f}/_{\partial z}\cdot\overline{\left({}^{\partial f}/_{\partial\bar z}\right)}\cdot(dz)^2\right]$ is not a multiple of $|dz|^2$ it must be zero, whence ${}^{\partial f}/_{\partial z}\cdot\overline{\left({}^{\partial f}/_{\partial\bar z}\right)}=0$. It follows that for each point z_0 we have ${}^{\partial f}/_{\partial z}(z_0)=0$ or ${}^{\partial f}/_{\partial \bar z}(z_0)=0$ (but not both, as $d_{z_0}f\neq 0$). Then

$$M=\left\{z_0:\frac{\partial f}{\partial z}(z_0)=0\right\}\bigcup\left\{z_0:\frac{\partial f}{\partial \bar z}(z_0)=0\right\}.$$

Since these two sets are closed and disjoint, one of them must be empty, so that f is holomorphic or anti-holomorphic in M.

The above calculations show as well that if $f : M \to N$ is holomorphic or anti-holomorphic then it is conformal. □

Let us consider the Riemann sphere $S^2 = \mathbb{CP}^1$ naturally identified with the set $\mathbb{C}\cup\{\infty\}$ (where $\infty = 0^{-1}$). We define the two classes of mappings of $\mathbb{CP}^1$ onto itself by

$$\text{homographies}: \qquad z\mapsto\frac{az+b}{cz+d}$$

$$\text{anti-homographies}: \qquad z\mapsto\frac{a\bar z+b}{c\bar z+d}$$

where $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ varies in $\mathrm{Gl}(2,\mathbb{C})$.

The following theorem settles the two-dimensional conformal geometry for the most important domains. We shall identify $\mathbb{R}^2$ with $\mathbb{C}$, in such a way that $\mathbb{R}^2$, D^2 and $\Pi^{2,+}$ are open subsets of $\mathbb{CP}^1$. If F is a set of mappings we denote by $c(F)$ the set $\{(z\mapsto\overline{f(z)}) : f\in F\}$ and by $-F$ the set $\{(z\mapsto -f(z)) : f\in F\}$.

All results we shall need from the theory of one complex variable can be found *e.g.* in [La] and [Na].

Theorem A.3.3. The group $\mathrm{Conf}^+(S^2)$ consists of all homographies, and the group $\mathrm{Conf}(S^2)$ consists of all homographies and anti-homographies. For $M = \mathbb{R}^2, D^2, \Pi^{2,+}$ we have

$$\mathrm{Conf}^+(M) = \{f|_M : f \in \mathrm{Conf}^+(S^2), f(M) = M\}$$
$$\mathrm{Conf}(M) = \{f|_M : f \in \mathrm{Conf}(S^2), f(M) = M\}.$$

In particular:

$$\mathrm{Conf}^+(\mathbb{C}) = \{(z \mapsto az + b) : a, b \in \mathbb{C}, a \neq 0\}$$
$$\mathrm{Conf}(\mathbb{C}) = \mathrm{Conf}^+(\mathbb{C}) \bigcup c(\mathrm{Conf}^+(\mathbb{C}))$$
$$\mathrm{Conf}^+(D^2) = \left\{\left(z \mapsto e^{i\theta} \cdot \frac{z-\alpha}{1-\overline{\alpha}z}\right) : \theta \in \mathbb{R}, \alpha \in D^2\right\}$$
$$\mathrm{Conf}(D^2) = \mathrm{Conf}^+(D^2) \bigcup c(\mathrm{Conf}^+(D^2))$$
$$\mathrm{Conf}^+(\Pi^{2,+}) = \left\{\left(z \mapsto \frac{az+b}{cz+d}\right) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}(2, \mathbb{R})\right\}$$
$$\mathrm{Conf}(\Pi^{2,+}) = \mathrm{Conf}^+(\Pi^{2,+}) \bigcup \left(-c(\mathrm{Conf}^+(\Pi^{2,+}))\right).$$

Proof. By Proposition A.3.2 we have to determine the set of holomorphisms and anti-holomorphisms of these complex surfaces. We shall refer only to holomorphisms; all the details for the case of anti-holomorphisms can be filled in as an exercise.

We begin with the explicit determination of the holomorphisms in all cases.

If $f : \mathbb{C} \to \mathbb{C}$ is a holomorphism then f cannot have an essential singularity at ∞ (otherwise, by Picard's theorem, it would not be one-to-one); the power series expansion of f at 0

$$f(z) = \sum_{n \geq 0} a_n \cdot z^n$$

coincides with the Laurent expansion of f at ∞, and hence it is finite. It follows that f is a polynomial, and bijectivity immediately implies that $f(z) = az + b$ with $a \neq 0$.

As for $\mathbb{CP}^1$, the set of all homographies is a group of holomorphisms of $\mathbb{CP}^1$. Conversely, since homographies operate transitively, given a holomorphism f we can find a homography ϕ with $(\phi \circ f)(\infty) = \infty$; it follows that $(\phi \circ f)$ is a holomorphism of $\mathbb{C}$, and hence it is a homography, which implies that f is a homography too.

By Schwarz's lemma the group of holomorphisms of D^2 keeping the origin fixed is given by rotations, and the proof works as above since the described set is a group of holomorphisms of D^2 containing rotations and operating transitively.

The determination of the group $\mathrm{Conf}^+(\Pi^{2,+})$ easily follows from that of $\mathrm{Conf}^+(D^2)$ via the Cayley transformation $z \mapsto {}^{(z-i)}/_{(z+i)}$, which maps $\Pi^{2,+}$ bi-holomorphically onto D^2.

Now, let $M \in \{\mathbb{C}, D^2, \Pi^{2,+}\}$; we are left to prove that

$$\mathrm{Conf}^+(M) = \{f|_M : f \in \mathrm{Conf}^+(S^2), f(M) = M\}.$$

If f is a homography and $f(M) = M$ the restriction of f to M is obviously a holomorphism of M. As for the converse, it easily follows from the determination of $\mathrm{Conf}^+(M)$ in the three cases that all its elements extend to homographies. □

As for completeness, we recall the usual representation of the group $\mathrm{Conf}^+(D^2)$ (see [Ve]). After defining

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \mathrm{SU}(1,1) = \{A \in \mathrm{Sl}(2, \mathbb{C}) : \ {}^t\overline{A}JA = J\}$$

it is checked that

$$\mathrm{SU}(1,1) = \left\{\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\right\}$$
$$\mathrm{Conf}^+(D^2) = \left\{\left(z \mapsto \frac{az+b}{cz+d}\right) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SU}(1,1)\right\} \cong {}^{\mathrm{SU}(1,1)}/_{\{\pm I\}}.$$

Proposition A.3.4. If we identify $\mathbb{CP}^1$ with $\mathbb{R}^2 \cup \{\infty\}$ then $\mathrm{Conf}(\mathbb{CP}^1)$ consists of all and only the mappings of the form

$$x \mapsto \lambda A i(x) + v$$

where $\lambda > 0$, $A \in \mathrm{O}(2)$, i is either the identity or an inversion and $v \in \mathbb{R}^2$.

Proof. Since the conjugation is an element of $\mathrm{O}(2)$ we consider an anti-homography

$$f : z \mapsto \frac{a\overline{z}+b}{c\overline{z}+d}$$

and we show that f can be written as $\lambda A i + v$. If $c = 0$ this fact is obvious. If $c \neq 0$ we have

$$\frac{a\overline{z}+b}{c\overline{z}+d} = \frac{a}{c} + \frac{{}^{(bc-ad)}/_{c^2}}{\overline{z} + {}^d/_c}.$$

Let ${}^{(bc-ad)}/_{c^2} = \lambda u$, with $\lambda > 0$ and $|u| = 1$ (hence $u \in \mathrm{O}(2)$); if we define i to be the inversion with respect to the sphere of centre $-\overline{{}^d/_c}$ and radius 1, we have

$$i(z) = -\overline{{}^d/_c} + \frac{z + \overline{{}^d/_c}}{\left|z + \overline{{}^d/_c}\right|^2} = -\overline{{}^d/_c} + \frac{1}{\overline{z} + {}^d/_c}$$

and hence

$$\frac{a\overline{z}+b}{c\overline{z}+d} = \lambda u i(z) + \lambda u \overline{{}^d/_c} + {}^a/_c.$$

A similar calculation proves that every mapping of the form $\lambda A i + v$ is a homography or an anti-homography. □

The following result could be proved as a corollary of A.3.4, but we shall prove it directly from A.3.3. In A.3.9 we shall check that a completely analogous statement holds for $n \geq 3$ too.

Theorem A.3.5. (1) $\mathrm{Conf}(D^2)$ consists of all and only the mappings of the form $x \mapsto Ai(x)$, where $A \in O(2)$ and i is either the identity or an inversion with respect to a sphere orthogonal to ∂D^2.
(2) $\mathrm{Conf}(\Pi^{2,+})$ consists of all and only the mappings of the form

$$x \mapsto \lambda \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} i(x) + \begin{pmatrix} b \\ 0 \end{pmatrix}$$

where $\lambda > 0$, $u \in O(1) = \{\pm 1\}$, i is either the identity or an inversion with respect to a sphere orthogonal to $\mathbb{R} \times \{0\}$ and $b \in \mathbb{R}$.

Proof. (1) By A.3.1 (4) and (5) every mapping of the form Ai belongs to $\mathrm{Conf}(D^2)$. As for the converse, we remark that the set of all the mappings of the required form is a group: hence, by A.3.3, since the conjugation and all rotations belongs to $O(2)$ we only have to check that for $\alpha \in D^2$ the function $z \mapsto {}^{(\bar{z} - \alpha)}/_{(1 - \bar{\alpha}\bar{z})}$ can be written as Ai. The sphere of centre ${}^1/_{\bar{\alpha}}$ and squared radius ${}^1/_{|\alpha|^2} - 1$ is orthogonal to ∂D^2; let i denote the inversion with respect to it; we have

$$\begin{aligned} i(z) &= \frac{1}{\bar{\alpha}} + \left(\frac{1}{|\alpha|^2} - 1\right) \frac{1}{\bar{z} - {}^1/_{\alpha}} = \\ &= \frac{1}{\bar{\alpha}} + \frac{1 - |\alpha|^2}{\bar{\alpha}} \cdot \frac{1}{\alpha\bar{z} - 1} = \frac{1}{\bar{\alpha}} \cdot \frac{\alpha\bar{z} - |\alpha|^2}{\alpha\bar{z} - 1} = -\frac{\alpha}{\bar{\alpha}} \cdot \frac{\bar{z} - \bar{\alpha}}{1 - \alpha\bar{z}} \\ \Rightarrow \frac{\bar{z} - \alpha}{1 - \bar{\alpha}\bar{z}} &= -\frac{\alpha}{\bar{\alpha}} \cdot i(z). \end{aligned}$$

(2) By A.3.1 (4) and (6)-i) every mapping of the described form belongs to $\mathrm{Conf}(\Pi^{2,+})$. As for the converse, we remark that the set of all the mappings of the required form is a group: then, by A.3.3, as the mapping $z \mapsto -\bar{z}$ is expressed by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, we only have to check that for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}(2, \mathbb{R})$ the mapping

$$z \mapsto -\frac{a\bar{z} + b}{c\bar{z} + d}$$

can be written in the required form. If $c = 0$ this is obvious. Otherwise we set $\mu = -{}^{(bc - ad)}/_{c^2} = {}^1/_{c^2} > 0$ and we consider the inversion i with respect to the sphere of centre $-{}^d/_c$ and radius 1 (which is orthogonal to $\mathbb{R} \times \{0\}$, since its centre lies on such a line); it is easily checked that

$$-\frac{a\bar{z} + b}{c\bar{z} + d} = \mu i(z) + \mu\frac{d}{c} - \frac{a}{c}$$

and the conclusion follows immediately. □

Remark A.3.6. Since translations, dilations and elements of $SO(n)$ preserve orientation, while elements of $O(n) \setminus SO(n)$ and inversions with respect to spheres reverse orientation, we have the following:
(1) $Ai \in \mathrm{Conf}(D^2)$ belongs to $\mathrm{Conf}^+(D^2)$ if and only if

$$[A \in SO(2), i = \text{identity}] \text{ or } [A \in O(2) \setminus SO(2), i = \text{inversion}].$$

(2) $\left(\lambda \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} i + \begin{pmatrix} b \\ 0 \end{pmatrix}\right) \in \mathrm{Conf}(\Pi^{2,+})$ belongs to $\mathrm{Conf}^+(\Pi^{2,+})$ if and only if $[u = 1, i = \text{identity}]$ or $[u = -1, i = \text{inversion}]$.

SECOND CASE: $n \geq 3$.

Our aim is to prove an analogue of Theorem A.3.5 for the n-dimensional ball and half-space. We shall use a much more general result due to Liouville, whose long proof we are going to present now. It is worth remarking that an analogue of Liouville's theorem in dimension two is false: we shall point out the steps where the assumption $n \geq 3$ is essential.

Theorem A.3.7 (Liouville). Every conformal diffeomorphism between two domains of $\mathbb{R}^n$ has the form

$$x \mapsto \lambda Ai(x) + b$$

where $\lambda > 0, A \in O(n), i$ is either the identity or an inversion and $b \in \mathbb{R}^n$.

Proof. If U, V are domains in $\mathbb{R}^n$ and $f : U \to V$ is a conformal diffeomorphism, we shall denote by $\mu_f \in C^\infty(U, \mathbb{R}_+)$ the coefficient of dilation of the metric, that is the function satisfying

$$\|d_x f(v)\| = \mu_f(x)\,\|v\| \qquad \forall x \in U, v \in \mathbb{R}^n.$$

We define ρ_f as ${}^1/_{\mu_f}$.

We shall say f is of type (a) if it is expressed as $\lambda A + b$ with $A \in O(n)$, $b \in \mathbb{R}^n$, and of type (b) if it is expressed as $\lambda Ai + b$, where $A \in O(n)$, $b \in \mathbb{R}^n$ and i is the inversion with respect to a sphere. The theorem can be re-phrased as follows: *every conformal diffeomorphism* $f : U \to V$ *is either of type* (a) *or of type* (b).

The proof is a straight-forward corollary of the following partial results:

Step 1.
i) f is of type (a) if and only if ρ_f is constant;
ii) f is of type (b) if and only if there exist $x_0 \in \mathbb{R}^n$ and $\eta \in \mathbb{R} \setminus \{0\}$ such that $\rho_f(x) = \eta\,\|x - x_0\|^2$.

Step 2. There exist $\eta, \tau \in \mathbb{R}$, $z \in \mathbb{R}^n$ such that $\rho_f(x) = \eta\,\|x\|^2 + \langle x|z\rangle + \tau$.

Step 3. If in step 2 it is $\eta \neq 0$ then for some $x_0 \in \mathbb{R}^n$ we have

$$\rho_f(x) = \eta\,\|x - x_0\|^2.$$

Step 4. In step 2 it cannot occur that $\eta = 0$ and $z \neq 0$.

According to Step 2, we shall say that f is of type I if $\eta = 0$ and $z = 0$, of type II if $\eta = 0$ and $z \neq 0$, of type III if $\eta \neq 0$. By step 1, if f is of type I then it is of type (a), and steps 3 and 4 can be respectively re-phrased as follows:

– if f is of type III then it is of type (b).

– f cannot be of type II.

We turn to the proof of these steps.

Proof of step 1. i) The "only if" part is obvious. As for the "if" part let us remark that

$$\tilde{f} = \rho_f \cdot f : U \to \rho_f \cdot V$$

is an isometry. If $x_0 \in U$ we set

$$\phi : U \to \mathbb{R}^n \qquad x \mapsto d_{x_0}\tilde{f}(x - x_0) + \tilde{f}(x_0);$$

since ϕ is obviously an isometry onto its range, by Proposition A.2.1 ϕ and $\tilde{f}$ coincide, whence f is of type (a).

ii) The "only if" part follows from the fact that the the coefficient of dilation of the metric for an inversion $i_{x_0,\alpha}$ in a point x is given by ${}^{\alpha}/_{\|x - x_0\|^2}$. As for the "if" part we remark that if $\rho_f(x) = \eta\,\|x - x_0\|^2$ and $i = i_{x_0,1/\eta}$ then $f \circ i$ is an isometry of $i(U)$ onto V with constant dilation coefficient, and the above argument applies.

Proof of step 2. We set $\rho = \rho_f$ and $\mu = \mu_f$. If $x \in U$ and $d_x^2\rho$ denotes the second differential of ρ in x, we shall prove the following facts:

(i) $d_x^2\rho(u, w) = 0$ if $u \perp w$;

(ii) $d_x^2\rho(u, w) = \eta(x)\langle u|w\rangle$ for some $\eta \in C^\infty(U)$;

(iii) η is a constant function;

Then the conclusion follows from the fact that the general solution of the differential equation $d_x^2\rho(u, w) = \eta\langle u|w\rangle$ has the form

$$\rho(x) = \frac{\eta}{2}\|x\|^2 + \langle x|z\rangle + \tau \qquad z \in \mathbb{R}^n, \tau \in \mathbb{R}.$$

(i) Let u, v, w be pairwise orthogonal vectors; we shall often use without mention the fact that they can be taken to be simultaneously non-zero. If we consider the partial derivative in direction w of the identity

$$\langle d_x f(u)|d_x f(v)\rangle = 0$$

we obtain

$$\langle d_x^2 f(u, w)|d_x f(v)\rangle = -\langle d_x f(u)|d_x^2 f(v, w)\rangle$$

If we allow u, v, w to vary (with the condition that they keep pairwise orthogonal), we obtain that the left hand side is symmetric in the pairs (u, w) and (v, w), and skew-symmetric in the pair (u, v), which implies that it is identically zero. Now, let us fix u and w. Since the image under $d_x f$ of the subspace orthogonal to u and w is the subspace orthogonal to $d_x f(u)$ and $d_x f(w)$, we deduce from above that for some real functions α and β depending on u and w

$$d_x^2 f(u, w) = \alpha(x) d_x f(u) + \beta(x) d_x f(w).$$

If we consider the partial derivative in direction w of the identity

$$\|d_x f(u)\|^2 = \mu(x)^2\|u\|^2$$

we obtain

$$\langle d_x^2 f(u, w)|d_x f(u)\rangle = \mu(x) d_x\mu(w)\|u\|^2$$

therefore

$$\alpha(x) = \frac{d_x\mu(w)}{\mu(x)}$$

and similarly

$$\beta(x) = \frac{d_x\mu(u)}{\mu(x)}.$$

Since $d_x\rho(z) = -\dfrac{d_x\mu(z)}{\mu(x)^2}$ we obtain that

$$\rho(x) d_x^2 f(u, w) + d_x\rho(w) d_x f(u) + d_x\rho(u) d_x f(w) = 0.$$

This identity holds for all $x \in U$ and $u, w \in \mathbb{R}^n$ with the only condition that $\langle u|w\rangle = 0$. If v is orthogonal to u and w and we take the partial derivative in direction v we obtain

$$d_x\rho(v) d_x^2 f(u, w) + \rho(x) d_x^3 f(u, w, v) + d_x^2\rho(w, v) d_x f(u) +$$
$$+ d_x\rho(w) d_x^2 f(u, v) + d_x^2\rho(u, v) d_x f(w) + d_x\rho(u) d_x^2 f(w, v) = 0.$$

The second, the fourth and the fifth terms are symmetric in the pair (u, v), and the same holds for the sum of the first and the sixth terms, so that the third term is symmetric in (u, v) too:

$$d_x^2\rho(w, v) d_x f(u) = d_x^2\rho(w, u) d_x f(v);$$

but $d_x f(u)$ and $d_x f(v)$ are mutually orthogonal, whence

$$d_x^2\rho(w, u) = 0 \qquad \forall x \in U \ w, u \in \mathbb{R}^n \ s.t.\ \langle w|u\rangle = 0.$$

(ii) For fixed x $d_x^2\rho$ is a symmetric bi-linear form on $\mathbb{R}^n$. By (i), if $\{e_1, ..., e_n\}$ is the canonical basis of $\mathbb{R}^n$, we have

$$d_x^2\rho(e_i, e_j) = k_i\delta_j^i \quad (k_i \in \mathbb{R}).$$

Moreover for $i \neq j$

$$0 = d_x^2\rho(e_i + e_j, e_i - e_j) = k_i - k_j \ \Rightarrow \ k_i = k_j$$

and then $d_x^2\rho$ must be a multiple of the scalar product. The dependence of the multiplying constant on x is obviously differentiable.

(iii) If we consider the partial derivative in an arbitrary direction v of the identity

$$d_x^2\rho(w, u) = \eta(x)\langle w|u\rangle$$

we obtain

$$d_x^3\rho(w,u,v) = d_x\eta(v)\langle w|u\rangle.$$

Since $d_x^3\rho$ is symmetric we have

$$\langle d_x\eta(v)w - d_x\eta(w)v|u\rangle = 0$$

which implies that $d_x\eta(v)w = d_x\eta(w)v$ and therefore $d_x\eta = 0$, *i.e.* η is a constant.

Proof of step 3. It is readily verified that

$$\eta\,\|x\|^2 + \langle x|z\rangle + \tau = \eta\,\|x - x_0\|^2 + \tau'$$

where $x_0 = -{}^{z}/_{2\eta}$, $\tau' = \tau - \eta\,\|x_0\|^2$. We must check that $\tau' = 0$.

We set $g = f^{-1}$ and we remark that g cannot be of type I (otherwise f would be of type I too). The set

$$\mathcal{F}_1 = \Big\{\{x\in U : \rho_f(x) = \lambda\} : \lambda > 0\Big\}$$

is a family of spheres centred at x_0 intersected with U, while the set

$$\mathcal{F}_2 = \Big\{\{y\in V : \rho_g(y) = \lambda\} : \lambda > 0\Big\}$$

is a family of spheres or hyperplanes intersected with V, according to the fact that g is of type III or II. Moreover, by the obvious relation

$$\rho_g(f(x)) = \rho_f(x)^{-1} \ \forall x\in U,$$

f maps $\mathcal{F}_1$ bijectively onto $\mathcal{F}_2$.

Since f is conformal, if γ is an arc in U orthogonal to all the elements of $\mathcal{F}_1$, then $f\circ\gamma$ is an arc in V orthogonal to all the elements of $\mathcal{F}_2$; such an arc can be re-parametrized as $t\mapsto y_0 + tu_2$, where

$$\begin{cases} \text{if } g \text{ is of type III, } y_0 \text{ is s.t. } \rho_g(y) = \eta'\|y - y_0\|^2 + \tau'' \text{ and } u_2 \neq 0 \\ \text{if } g \text{ is of type II, } u_2 \text{ is s.t. } \rho_g(y) = \langle y|u_2\rangle + \tau'' \text{ and } y_0\in\mathbb{R}^n. \end{cases}$$

Let γ have the form $\gamma(t) = x_0 + tu_1$, $|t - t_0| < \varepsilon$; then we have

$$(f\circ\gamma)(t) = y_0 + \phi(t)u_2,$$

where y_0 and u_2 are as above and ϕ is a diffeomorphism onto an open interval in $\mathbb{R}$. We have

$$|\dot\phi(t)|\cdot\|u_2\| = \|(f\circ\gamma)'(t)\| = \|d_{\gamma(t)}f(u_1)\| = \frac{\|u_1\|}{\rho_f(\gamma(t))}$$

$$\Rightarrow\ \dot\phi(t) = \pm\frac{\|u_1\|}{\|u_2\|\cdot(\eta t^2\|u_1\|^2 + \tau')}.$$

Now, if we assume by contradiction that $\tau'\neq 0$, we can find $\lambda\in\mathbb{C}\setminus\{0\}$ (real or purely imaginary) and $k\in\mathbb{R}\setminus\{0\}$ such that

$$\dot\phi(t) = \frac{k}{t^2 - \lambda^2}.$$

The image of the interval $(t_0 - \varepsilon, t_0 + \varepsilon)$ under the transformation

$$t\mapsto {}^{(t-\lambda)}/_{(t+\lambda)}$$

is connected and simply connected and it does not contain 0, so that we can find a holomorphic determination log of the logarithm function defined on a neighborhood of it. The function

$$\psi : (t_0 - \varepsilon, t_0 + \varepsilon)\ni t\mapsto \frac{k}{2\lambda}\log\left(\frac{t-\lambda}{t+\lambda}\right)\in\mathbb{C}$$

is well-defined and differentiable, and $\dot\psi = \dot\phi$, which implies that for some $k_1\in\mathbb{C}$

$$\phi(t) = \frac{k}{2\lambda}\log\left(\frac{t-\lambda}{t+\lambda}\right) + k_1.$$

According to the fact that g is of type III or II, the condition

$$\rho_g(f(\gamma(t)))\cdot\rho_f(\gamma(t)) = 1$$

can be re-written respectively as

$$(*)\qquad \begin{aligned} &(\eta'\phi(t)^2\|u_2\|^2 + \tau'')(\eta t^2\|u_1\|^2 + \tau') = 1 \\ &(\phi(t)\|u_2\|^2 + \tau''')(\eta t^2\|u_1\|^2 + \tau') = 1. \end{aligned}$$

Let us remark that it is known by now that one of these relations is true for $t\in(t_0 - \varepsilon, t_0 + \varepsilon)$. However, by the explicit expression of ϕ, if Ω is an open subset of $\mathbb{C}$ containing $(t_0 - \varepsilon, t_0 + \varepsilon)$ and the image of Ω under the mapping $t\mapsto {}^{(t-\lambda)}/_{(t+\lambda)}$ is connected and simply connected and does not contain 0, the definition of ϕ can be extended holomorphically to Ω, and hence the above relation holds for t in Ω. In particular we can choose Ω in such a way that for some $w\in\mathbb{C}\setminus\{0\}$ and $\delta > 0$ it contains the segment $\{\lambda + sw : 0 < s < \delta\}$. By the choice of λ we have $\eta\lambda^2\|u_1\|^2 + \tau' = 0$, and then relations $(*)$ can be re-written for $t = \lambda + sw$ as

$$(\eta'\phi(\lambda + sw)^2\|u_2\|^2 + \tau'')(2\lambda + ws)\eta w\|u_1\|^2\cdot s = 1$$

$$(\phi(\lambda + sw)\|u_2\|^2 + \tau''')(2\lambda + ws)\eta w\|u_1\|^2\cdot s = 1.$$

But now we have that

$$\lim_{s\to 0}\phi(\lambda + sw)s = \lim_{s\to 0}\phi(\lambda + sw)^2 s = 0$$

and hence both the above relations imply the contradiction 0=1.

Proof of step 4. The argument is completely analogous to the one presented for step 3, so we shall work out calculations without comments; we assume by contradiction that $\eta = 0$ and $z\neq 0$.

$$\rho_f(x) = \langle x|u_1\rangle + \tau. \qquad \gamma(t) = x_0 + tu_1 \ \Rightarrow\ (f\circ\gamma)(t) = y_0 + \phi(t)u_2.$$

$$\begin{cases} \rho_f(\gamma(t)) = \rho_f(x_0 + tu_1) = t\|u_1\|^2 + \tau', \\ |\dot\phi(t)|\cdot\|u_2\| = \|(f\circ\gamma)'(t)\| = \dfrac{\|u_1\|}{\rho_f(\gamma(t))} \end{cases}$$

$$\Rightarrow \dot{\phi}(t) = \pm \frac{\|u_1\|}{\|u_2\| \cdot (t\|u_1\|^2 + \tau')}$$

$\exists k \in \mathbb{R} \setminus \{0\}$ such that $\dot{\phi}(t) = \frac{k}{t+k'}$ $(k' = \tau'/_{\|u_1\|^2})$

$$\Rightarrow \phi(t) = k \log(t + k') + k''.$$

$$\begin{cases} (\eta'\phi(t)^2\|u_2\|^2 + \tau'') \cdot (t\|u_1\|^2 + \tau') = 1 & (g \text{ type III}) \\ (\phi(t)\|u_2\|^2 + \tau''') \cdot (t\|u_1\|^2 + \tau') = 1 & (g \text{ type II}) \end{cases}$$

$$\Rightarrow \begin{cases} (\eta'\phi(s-k')^2\|u_2\|^2 + \tau'') \cdot \|u_1\|^2 \cdot s = 1 \\ (\phi(t)\|u_2\|^2 + \tau''') \cdot \|u_1\|^2 \cdot s = 1 \end{cases}$$

$$\lim_{s \to 0} \phi(s - k')s = \lim_{s \to 0} \phi(s - k')^2 s = 0 \Rightarrow 0 = 1. \text{ Absurd.}$$

The proof of Liouville's theorem is now complete. □

It is quite interesting to remark that the assumption that $n \geq 3$ was used only in the proof of Step 2-(i); however, it can be easily checked that the assumption cannot be dropped. For instance, on the unit disc D^2 of $\mathbb{C}$ every one-to-one holomorphic mapping is a conformal diffeomorphism onto its range; and plenty of holomorphic functions are one-to-one on the disc (*e.g.* if $p \in \mathbb{C}[x]$ and $p'(0) \neq 0$, if $\varepsilon > 0$ is small enough the function $z \mapsto p(\varepsilon z)$ is one-to-one on the disc).

Moreover the Riemann mapping theorem (see [Na]) implies that every simply connected proper domain of $\mathbb{C}$ is conformally equivalent to D^2, while A.3.7 implies that for $n \geq 3$ only open balls and open subspaces are conformally equivalent to D^n. This fact is the first feature of a phenomenon of *rigidity* for the case $n \geq 3$ we shall discuss in Chapt. C.

In the following two results we shall check that in spite of the differences between the two cases $n = 2$ and $n \geq 3$, the determination of the groups $\mathrm{Conf}(S^n)$, $\mathrm{Conf}(D^n)$ and $\mathrm{Conf}(\Pi^{n,+})$ given for $n = 2$ in A.3.4 and A.3.5 can be generalized word-by-word to the case $n \geq 3$.

Corollary A.3.8. $\mathrm{Conf}(S^n)$ consists of all and only the mappings of the form

$$x \mapsto \lambda A i(x) + b$$

where $\lambda > 0$, $A \in \mathrm{O}(n)$, i is either the identity or the inversion with respect to a sphere and $b \in \mathbb{R}^n$.

If M and N are domains in S^n then

$$\mathrm{Conf}(M, N) = \{f|_M : f \in \mathrm{Conf}(S^n), f(M) = N\}.$$

Proof. The mappings of the form $\lambda A i + b$ constitute a group of conformal diffeomorphisms operating transitively on S^n and containing the isotropy group of ∞ (that is, $\mathrm{Conf}(\mathbb{R}^n)$), and therefore this group is $\mathrm{Conf}(S^n)$.

The second assertion is a straight-forward consequence of A.3.7. □

Theorem A.3.9. Let $n \geq 2$.

(1) $\mathrm{Conf}(D^n)$ consists of all and only the mappings of the form

$$x \mapsto A i(x),$$

where $A \in \mathrm{O}(n)$ and i is either the identity or an inversion with respect to a sphere orthogonal to ∂D^n.

(2) $\mathrm{Conf}(\Pi^{n,+})$ consists of all and only the mappings of the form

$$x \mapsto \lambda \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} i(x) + \begin{pmatrix} b \\ 0 \end{pmatrix}$$

where $\lambda > 0$, $A \in \mathrm{O}(n-1)$, i is either the identity or an inversion with respect to a sphere orthogonal to $\mathbb{R}^{n-1} \times \{0\}$ and $b \in \mathbb{R}^{n-1}$.

Proof. By A.3.5 we only have to consider the case $n \geq 3$.

(1) By A.3.1 the set of all the mappings of the required form is a group of conformal diffeomorphisms of D^n. As for the converse, let $f \in \mathrm{Conf}(D^n)$. Assume first that $f = \lambda A + b$; since the ball of centre b and radius λ must be D^n, it follows that $b = 0$ and $\lambda = 1$, whence f has the required form. If $f = \lambda A i + b$, with $i = i_{x_0,r}$, then certainly $x_0 \notin \overline{D^n}$ (otherwise we would have $\infty \in \overline{f(D^n)} = \overline{D^n}$, which is false). Let us consider the inversion j with respect to the sphere centred at x_0 with radius $\sqrt{\|x_0\|^2 - 1}$ (which is orthogonal to ∂D^n); then $f \circ j \in \mathrm{Conf}(D^n)$ and it has the form, $\lambda' A' + b'$, and the conclusion follows from the first part.

(2) By A.3.1 the set of all the mappings of the required form is a group of conformal diffeomorphisms of $\Pi^{n,+}$. As for the converse, let $f \in \mathrm{Conf}(\Pi^{n,+})$. Assume first that $f = \lambda A + b$; since 0 belongs to the boundary of $\Pi^{n,+}$, the same must hold for $f(0) = b$, so the last coordinate of b must be 0. Then $A = \lambda^{-1}(f - b) \in \mathrm{Conf}(\Pi^{n,+})$. If for some $j < n$ the element a_{nj} on the n-th row and j-th column of A does not vanish, the image under A of the j-th element e_j of the canonical basis of $\mathbb{R}^n$ does not belong to the boundary of $\Pi^{n,+}$, while e_j does, and this is absurd. Since the same argument works for $A^{-1} = {}^tA$, A must have the form $\begin{pmatrix} B & 0 \\ 0 & w \end{pmatrix}$, and then we obviously have that $B \in \mathrm{O}(n-1)$ and $w = 1$, whence f has the required form. Now, if $f = \lambda A i + b$, where i is the inversion with respect to a sphere centred at a point x_0, since $f(x_0) = \infty$ belongs to the boundary of $\Pi^{n,+}$, the same must hold for x_0, *i.e.* $x_0 \in \mathbb{R}^{n-1} \times \{0\}$, and then every sphere centred at x_0 is orthogonal to $\mathbb{R}^{n-1} \times \{0\}$. The conclusion follows from the fact that $f \circ i \in \mathrm{Conf}(\Pi^{n,+})$ has the form $\lambda' A' + b'$. □

Remark A.3.10. Everything we said in A.3.6 about conformal diffeomorphisms preserving or reversing orientation can be repeated word-by-word in the general case, so that in particular:

(1) $Ai \in \mathrm{Conf}(D^n)$ preserves the orientation if and only if

$$[A \in \mathrm{SO}(n), i = \mathrm{id}] \text{ or } [A \notin \mathrm{SO}(n), i \neq id].$$

(2) $\lambda\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} i + \begin{pmatrix} b \\ 0 \end{pmatrix} \in \mathrm{Conf}(\mathbb{H}^{n,+})$ preserves the orientation if and only if $[A \in \mathrm{SO}(n-1), i = \mathrm{id}]$ or $[A \notin \mathrm{SO}(n-1), i \neq \mathrm{id}]$.

We conclude this section with a technical result we shall need in the sequel:

Lemma A.3.11. Let $n \geq 2$. If $\phi \in \mathrm{Conf}(S^n)$ is not the identity and there exists a submanifold N of S^n of codimension 1 such that ϕ is the identity on N, then one (and only one) of the following facts is verified:
(a) N is contained in a sphere, and ϕ is the inversion with respect to this sphere;
(b) N is contained in a hyperplane, and ϕ is the reflection with respect to this hyperplane.

Proof. By A.3.4 and A.3.8 we have the following possibilities:
(a) $\phi = \lambda A + b$;
(b) $\phi = \lambda A i + b$, where i is an inversion.
We consider them separately.

(a) Let $x \in N$, $x \neq \infty$; since $d_x\phi|_{T_x N} = \mathrm{id}$ it must be $\lambda = 1$, hence ϕ is an isometry of $\mathbb{R}^n$. If $\{v_1, ..., v_{n-1}\}$ is an orthonormal basis of $T_x N$, we can complete it with a vector v_n to an orthonormal basis of $\mathbb{R}^n$. If $d_x\phi(v_n) = -v_n$ the reflection with respect to $T_x N$ is an isometry of $\mathbb{R}^n$ which coincides up to first order with ϕ in x, and hence it coincides with ϕ; since ϕ is the identity only on $T_x N$, obviously $N \subseteq T_x N$. The condition $d_x\phi(v_n) = v_n$ would imply that ϕ is the identity, and this is absurd.

(b) Let $i = i_{x_0,\alpha}$. For $x \in N$ and $v \in T_x N$ we have

$$\|v\| = \|d_x\phi(v)\| = \frac{\lambda\alpha}{\|x - x_0\|^2} \cdot \|v\| \;\Rightarrow\; \|x - x_0\|^2 = \lambda\alpha.$$

If we set $\beta = \lambda\alpha$ we have $N \subseteq M(x_0, \beta)$. Moreover $\phi \circ i_{x_0,\beta}$ is of type (a) and it is the identity on N; then it must be necessarily the identity, otherwise N would be contained in a hyperplane too, and this is absurd since the intersection of a sphere and a hyperplane has co-dimension at least 2. □

A.4 Isometries of Hyperbolic Space: Disc and Half-space Models

In this section the results of the long parenthesis about conformal geometry are used for the determination of the groups $\mathcal{I}(\mathbb{D}^n)$ and $\mathcal{I}(\mathbb{H}^{n,+})$.

Theorem A.4.1. $\mathcal{I}(\mathbb{D}^n) = \mathrm{Conf}(D^n)$, $\mathcal{I}^+(\mathbb{D}^n) = \mathrm{Conf}^+(D^n)$. In particular, these groups operate transitively on $\mathbb{D}^n$.

Proof. We start by proving that the restriction of the stereographic projection $p : \mathbb{I}^n \to D^n$ used for the definition of $\mathbb{D}^n$ is conformal ($\mathbb{I}^n$ is endowed with the hyperbolic metric and D^n with the Euclidean one). We shall denote by $\langle .|. \rangle_{(n,1)}$ the standard bi-linear symmetric form on $\mathbb{R}^{n+1}$, and by $\langle .|. \rangle$ the usual scalar product in $\mathbb{R}^n$. Since $p(x,t) = {}^x/_{(1+t)}$, we have

$$d_{(x,t)}p(y,s) = \frac{y}{1+t} - \frac{sx}{(1+t)^2};$$

for $(x,t) \in \mathbb{I}^n$, $(y,s), (z,r) \in T_{(x,t)}\mathbb{I}^n$ we have

$$\langle x|x \rangle = t^2 - 1 \qquad \langle x|y \rangle = st \qquad \langle x|z \rangle = rt$$

and hence

$$\langle d_{(x,t)}(y,s) | d_{(x,t)}(z,r) \rangle = \langle {}^y/_{1+t} - {}^{sx}/_{(1+t)^2} | {}^z/_{1+t} - {}^{rx}/_{(1+t)^2} \rangle =$$
$$= \frac{\langle y|z \rangle}{(1+t)^2} - \frac{2rst}{(1+t)^3} + \frac{rs(t^2-1)}{(1+t)^4} = \frac{1}{(1+t)^2}\left(\langle y|z \rangle - rs\right) =$$
$$= \frac{\langle (y,s)|(z,r) \rangle_{(n,1)}}{(1+t)^2}.$$

We have proved our assertion.

Let $\phi \in \mathcal{I}(\mathbb{D}^n)$; since by definition p is an isometry we have

$$p^{-1} \circ \phi \circ p \in \mathcal{I}(\mathbb{I}^n) \subseteq \mathrm{Conf}(\mathbb{I}^n)$$

and since p is conformal from $\mathbb{I}^n$ to D^n we obtain $\phi \in \mathrm{Conf}(D^n)$. Inclusion $\subseteq$ is proved.

As for the converse, using A.3.9, we shall prove that all the elements of $\mathrm{O}(n)$ and all the inversions with respect to spheres orthogonal to ∂D^n belong to $\mathcal{I}(\mathbb{D}^n)$. The first fact is easy: if $A \in \mathrm{O}(n)$, then

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{O}(\mathrm{I}_n) = \mathcal{I}(\mathbb{I}^n) \qquad p \circ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \circ p^{-1} = A.$$

For the second fact we remark that p^{-1} is expressed by

$$p^{-1} : x \mapsto \frac{1}{1 - \|x\|^2}\begin{pmatrix} 2x \\ 1 + \|x\|^2 \end{pmatrix};$$

if $(y,t) \in \mathbb{R}^{n+1}$ and $\langle (y,t)|(y,t) \rangle_{(n,1)} = 1$ the set

$$N = \left\{ x \in D^n : \langle p^{-1}(x) | (y,t) \rangle_{(n,1)} = 0 \right\}$$

is given by the equation

$$2\langle x|y \rangle = (1 + \|x\|^2)\sqrt{\|y\|^2 - 1}$$

i.e., if we set $w = {}^y/_{\sqrt{\|y\|^2 - 1}}$,

$$2\langle x|w \rangle = 1 + \|x\|^2 \;\Leftrightarrow\; \|x - w\|^2 = \|w\|^2 - 1$$

and hence N is the intersection of D^n with the sphere S_w of centre w and radius $\sqrt{\|w\|^2 - 1}$, which is orthogonal to ∂D^n.

If $\phi \in \mathcal{I}(\mathbb{I}^n)$ is the reflection parallel to (y,t), then $p \circ \phi \circ p^{-1}$ is a conformal diffeomorphism of D^n different from the identity and such that it is the identity on N; by A.3.5 and A.3.9 it extends to a conformal diffeomorphism of S^n; it follows from A.3.11 that $p \circ \phi \circ p^{-1}$ is the inversion with respect to S_w, and hence the latter belongs to $\mathcal{I}(\mathbb{D}^n)$. For the conclusion we only have to remark that if $w \notin D^n$ it is always possible to find $(y,t) \in \mathbb{R}^{n+1}$ such that $\langle (y,t) | (y,t) \rangle_{(n,1)} = 1$ and $w = {}^{y}/_{t}$.

The case of orientation-preserving isometries is now straight-forward. □

Theorem A.4.2. $\mathcal{I}(\mathbb{\Pi}^{n,+}) = \mathrm{Conf}(\Pi^{n,+})$,
$$\mathcal{I}^+(\mathbb{\Pi}^{n,+}) = \mathrm{Conf}^+(\Pi^{n,+}).$$

Proof. As we remarked in Sect. A.1, the diffeomorphism $i : D^n \to \Pi^{n,+}$ used for the definition of the hyperbolic structure on $\Pi^{n,+}$ is the inversion with respect to the sphere of centre $(0, ..., 0, -1)$ and radius $\sqrt{2}$, so it is conformal, and the conclusion follows immediately from A.4.1. □

Using A.4.1 and A.4.2 the hyperbolic metrics on $\mathbb{D}^n$ and $\mathbb{\Pi}^{n,+}$ can be explicitly computed. Since $\mathbb{D}^n$ and $\mathbb{\Pi}^{n,+}$ are open subsets of $\mathbb{R}^n$, their tangent bundles are canonically identified with $\mathbb{D}^n \times \mathbb{R}^n$ and $\mathbb{\Pi}^{n,+} \times \mathbb{R}^n$ respectively.

Theorem A.4.3. For $x \in \mathbb{D}^n$, $(y,t) \in \mathbb{\Pi}^{n,+}$ and $v \in \mathbb{R}^n$ the metrics are explicitly given by
$$ds_x^2(v) = \left(\frac{2}{1 - \|x\|^2}\right)^2 \|v\|^2$$
$$ds_{(y,t)}^2(v) = \frac{\|v\|^2}{t^2}.$$

Proof. The differential of the mapping $p : \mathbb{I}^n \to \mathbb{D}^n$ at the point $(0, ..., 0, 1)$ is half the identity on $\mathbb{R}^n \times \{0\} = T_{(0,1)}\mathbb{I}^n$, and the restriction of $\langle . | . \rangle_{(n,1)}$ to $\mathbb{R}^n \times \{0\}$ is the standard scalar product, so that
$$ds_0^2(v) = 4\|v\|^2.$$

For $x \in \mathbb{D}^n$ the inversion with respect to the sphere orthogonal to $\partial\mathbb{D}^n$ of centre ${}^{x}/_{\|x\|^2}$ is an isometry of $\mathbb{D}^n$ mapping 0 in x, and its differential in 0 is $(1 - \|x\|^2)$ times an orthogonal operator (compare A.3.1 (4)), and this implies that
$$ds_x^2(v) = \left(\frac{2}{1 - \|x\|^2}\right)^2 \|v\|^2.$$

Similarly, the differential of the inversion $i : \mathbb{D}^n \to \mathbb{\Pi}^{n,+}$ at the point 0 is twice an orthogonal operator, so that
$$ds_{(0,1)}^2(v) = \|v\|^2.$$

Since horizontal translations are isometries we only have to remark that the differential in $(0,1)$ of the dilation of coefficient $t > 0$ is t times the identity, and hence
$$ds_{(y,t)}^2(v) = \frac{\|v\|^2}{t^2}.$$
□

Corollary A.4.4. At any point of $\mathbb{D}^n$ and $\mathbb{\Pi}^{n,+}$ the hyperbolic metric is a positive mutiple of the Euclidean one.

Remark A.4.5. By A.4.4 the notion of conformal diffeomorphism on $\mathbb{D}^n$ and $\mathbb{\Pi}^{n,+}$ is the same if we consider the Euclidean metric and the hyperbolic one. Using A.4.1 and A.4.2, this implies that all conformal diffeomorphisms of $\mathbb{D}^n$ and $\mathbb{\Pi}^{n,+}$ with respect to the hyperbolic metric are isometries. This fact does not depend on the concrete representation of $\mathbb{H}^n$ (only the hyperbolic metric is used), so that we have the following proposition: *a sufficient condition for a diffeomorphism of* $\mathbb{H}^n$ *to preserve lengths is that it preserves angles* (the converse being true for all Riemannian manifolds). We can interprete heuristically this fact in the following way: *a unit of measure is intrinsically defined on* $\mathbb{H}^n$*, and it cannot be changed.* We shall prove other facts explaining this assertion.

A.5 Geodesics, Hyperbolic Subspaces and Miscellaneous Facts

We start with the hyperboloid model.

Proposition A.5.1. If $x \in \mathbb{I}^n$, $y \in T_x\mathbb{I}^n$, $\langle y|y \rangle_{(n,1)} = 1$ the geodesic starting at x with velocity y is given by
$$\mathbb{R} \ni t \mapsto \cosh(t) \cdot x + \sinh(t) \cdot y.$$

In particular as a set it is given by the intersection of $\mathbb{I}^n$ with the linear subspace of $\mathbb{R}^{n+1}$ generated by x an y.

Proof. Let W be the plane generated by x and y, and let ω be the maximal geodesic starting at x with velocity y; we shall confuse ω with its support. Let $\phi \in \mathrm{O}(\mathrm{I}_n)$ be defined by $\phi|_W = \mathrm{id}$ and $\phi|_{W^\perp} = -\mathrm{id}$. Since $\phi(x) = x$ and $d_x\phi(y) = y$, ω is ϕ-invariant, and therefore $\omega \subseteq W \cap \mathbb{I}^n$. Moreover it is easily checked that the mapping
$$\mathbb{R} \ni t \mapsto \cosh(t) \cdot x + \sinh(t) \cdot y$$
gives a parametrization of $W \cap \mathbb{I}^n$ with velocity of length identically 1, and therefore it coincides with ω. □

By the above result every geodesic in $\mathbb{H}^n$ is defined on the whole real line, and therefore the Hopf-Rinow theorem (see [He]) yields the following:

Corollary A.5.2. $\mathbb{H}^n$ is a complete Riemannian manifold.

An easy argument proves the following further consequence of A.5.1:

Corollary A.5.3. There exists one and only one geodesic line passing through any two different points of $\mathbb{H}^n$.

As in A.2.5 we record an analogy between $\mathbb{H}^n$ and S^n. We omit the proof since it works just like the one presented above for A.5.1.

Proposition A.5.4. If $x \in S^n$, $y \in T_x S^n$, $\langle y|y \rangle = 1$ the geodesic starting at x with velocity y is given by

$$\mathbb{R} \ni t \to \cos(t) \cdot x + \sin(t) \cdot y.$$

In particular as a set it is given by the intersection of S^n with the linear subspace of $\mathbb{R}^{n+1}$ generated by x and y.

We shall say a subset N of $\mathbb{H}^n$ is a hyperbolic subspace if it contains the entire geodesic passing through any two of its points. Remark that points and entire geodesics are hyperbolic subspaces. Proposition A.5.1 imply the following:

Corollary A.5.5. $N \subset \mathbb{I}^n$ is a hyperbolic subspace if and only if it is the intersection of $\mathbb{I}^n$ with a linear subspace of $\mathbb{R}^{n+1}$. In particular hyperbolic subspaces are submanifolds of $\mathbb{I}^n$, and hence their dimension is well-defined.

We consider now the other models of $\mathbb{H}^n$. Before stating the result we introduce some terminology.

– We shall say an affine subspace Y of $\mathbb{R}^n$ is vertical if it has the form $Y' + \mathbb{R}\, e_n$, where Y' is an affine subspace of $\mathbb{R}^{n-1} \times \{0\}$ and $e_n = (0, ...0, 1)$.

– From now on we shall allow a sphere in $\mathbb{R}^n$ to have dimension lower than $n-1$; this is obtained simply by considering the intersection of an $(n-1)$-dimensional sphere with an affine subspace passing through its centre. However, when speaking of inversion with respect to a sphere, we shall always mean that the sphere has maximal dimension.

– Let M_1 and M_2 be spheres or affine subspaces (or parts of) in $\mathbb{R}^n$, and let m_1, m_2 be their respective dimensions. We shall say that M_1 and M_2 are orthogonal if for each $x \in M_1 \cap M_2$ the linear space $W = T_x M_1 \cap T_x M_2$ has dimension $\max\{0, m_1 + m_2 - n\}$ and the orthogonal complements of W in $T_x M_1$ and $T_x M_2$ are orthogonal to each other.

Proposition A.5.6. (1) $N \subset \mathbb{D}^n$ is a hyperbolic subspace if and only if it is the intersection of $\mathbb{D}^n$ with a linear subspace of $\mathbb{R}^n$ or with a sphere orthogonal to $\partial\mathbb{D}^n$. In particular geodesics are obtained by parametrization of diameters of $\mathbb{D}^n$ and circles orthogonal to $\partial\mathbb{D}^n$.
(2) $N \subset \Pi^{n,+}$ is a hyperbolic subspace if and only if it is the intersection of $\Pi^{n,+}$ with an affine vertical subspace or with a sphere orthogonal to $\mathbb{R}^{n-1} \times \{0\}$. In particular geodesics are obtained by parametrization of vertical lines and circles orthogonal to $\mathbb{R}^{n-1} \times \{0\}$.

Proof. It is easily established that the image under the stereographic projection $p : \mathbb{I}^n \to \mathbb{D}^n$ of a linear subspace W passing through $(0, ..., 0, 1)$ is the intersection of $\mathbb{D}^n$ with a linear subspace of $\mathbb{R}^n$. Hence the hyperbolic subspaces of $\mathbb{D}^n$ passing through 0 are the intersections of $\mathbb{D}^n$ with the linear subspaces of $\mathbb{R}^n$. If $x \in \mathbb{D}^n$ the inversion i with respect to the sphere of centre $x/\|x\|^2$ and squared radius $1/\|x\|^2 - 1$ is an isometry of $\mathbb{D}^n$ and it maps 0 to x; this implies that it maps the set of all hyperbolic subspaces through 0 onto the set of all hyperbolic subspaces through x. Now, if Y is a linear subspace of dimension p of $\mathbb{R}^n$, we have two possibilities:

(i) $x \in Y$; in this case $i(Y) = Y$.

(ii) $x \notin Y$. If we consider the subspace X generated by Y and x, X is i-invariant and hence $i(Y) \subset X$. Since Y is a hyperplane in X, Proposition A.3.1 (6) implies that $i(Y)$ is a sphere in X, hence $i(Y)$ is a p-dimensional sphere in $\mathbb{R}^n$. Since i is conformal and Y is orthogonal to $\partial\mathbb{D}^n$, $i(Y)$ is orthogonal to $\partial\mathbb{D}^n$ too.

It is easily verified with the same method that i maps linear subspaces and spheres orthogonal to $\partial\mathbb{D}^n$ passing through x onto linear subspaces, and hence (1) is proved.

(2) The case of the half-space is a direct consequence of the previous one. In fact the mapping $i : \mathbb{D}^n \to \Pi^{n,+}$ used for the definition is an inversion, and hence by the above argument it maps the set of spheres and affine subspaces onto itself; moreover it preserves orthogonality and the conclusion follows at once. □

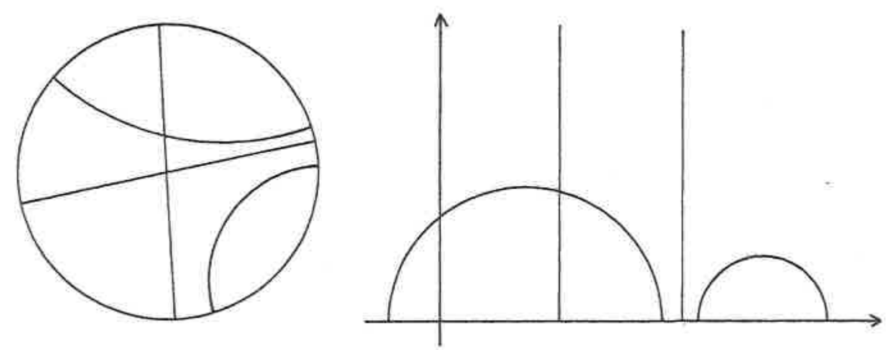

Fig. A.3. Geodesics in the two-dimensional disc and half-space

Corollary A.5.7. A p-dimensional hyperbolic subspace in $\mathbb{H}^n$ is isometrically diffeomorphic to $\mathbb{H}^p$.

Proof. We consider the disc model and assume that the hyperbolic subspace contains 0. By A.5.6 it is a p-dimensional disc, and by A.4.3 it inherits the same metric as that of $\mathbb{D}^p$. □

Proposition A.5.6 allows us to compute explicitly the hyperbolic distance in $\mathbb{D}^n$ and $\Pi^{n,+}$. We shall denote by th the hyperbolic tangent function and by ath its inverse function.

Corollary A.5.8. (1) If $x, y \in \mathbb{D}^n$ we have

$$d(x,y) = 2\operatorname{ath}\left(\frac{\|x-y\|}{\left(1-2\langle x|y\rangle + \|x\|^2\cdot\|y\|^2\right)^{1/2}}\right).$$

(2) If $(x,t),(y,s) \in \Pi^{n,+}$ we have

$$d\big((x,t),(y,s)\big) = 2\operatorname{ath}\left(\frac{\|x-y\|^2+(t-s)^2}{\|x-y\|^2+(t+s)^2}\right)^{1/2}.$$

Proof. (1) Let $v \in \mathbb{R}^n$, $\|v\| = 1$. A parametrization of the diameter determined by v is given by

$$\gamma : \mathbb{R} \ni t \to \operatorname{th}(t/2)\cdot v$$

and it is a straight-forward computation that

$$ds^2_{\gamma(t)}(\dot\gamma(t)) = 1.$$

It follows that for $x \in \mathbb{D}^n$ we have

$$d(0,x) = 2\operatorname{ath}\|x\|.$$

Now, let $x, y \in \mathbb{D}^n$. The inversion $i \in \mathcal{I}(\mathbb{D}^n)$ with respect to the sphere centred at $x/_{\|x\|^2}$ with squared radius $1/_{\|x\|^2} - 1$ can be written explicitly as

$$i(z) = (1-\|x\|^2)\frac{z\|x\|^2 - x}{\big\|z\|x\|^2 - x\big\|^2} + \frac{x}{\|x\|^2}.$$

Since $i(x) = 0$ we have

$$d(x,y) = d(0, i(y)) = 2\operatorname{ath}\|i(y)\|$$

and an easy calculation completes the proof.

(2) We only have to consider the inversion mapping $\mathbb{D}^n$ isometrically onto $\Pi^{n,+}$: explicit computations will be omitted. □

Remark A.5.9. It is well known (see [Si] or [Ve]) that a differential metric on D^2 with respect to which holomorphisms are isometries must be a multiple of the Poincaré one $\omega(z) = 4/_{|z|^2}$: by A.3.2 and A.4.1 this is the case for the hyperbolic metric. Proposition A.4.3 proves that in fact the hyperbolic metric and the Poincaré one coincide, and hence hyperbolic two-space is nothing but the Poincaré disc. The above calculation of the hyperbolic distance is consistent with this fact: it is well-known that the Poincaré distance is given by

$$d(w,z) = 2\operatorname{ath}\left|\frac{w-z}{1-w\,\overline{z}}\right|$$

and our formula is a natural generalization of this one.

We shall discuss now the notion of boundary of hyperbolic space. The symbol $\partial\mathbb{D}^n$ was already used to denote the boundary of D^n in $\mathbb{R}^n$, *i.e.* the sphere S^{n-1}, and similarly we can define $\partial\Pi^{n,+}$ as the boundary of $\Pi^{n,+}$ in S^n, *i.e.* $(\mathbb{R}^{n-1}\times\{0\})\cup\{\infty\}$. However a concrete representation of hyperbolic space is essential for these definitions. We shall prove now that the notion of boundary is intrinsically defined.

Consider the set $\mathcal{S}$ of all geodesic closed half-lines in $\mathbb{H}^n$ parametrized by arc length on $[0,\infty)$, and define an equivalence relation R on $\mathcal{S}$ in the following way:

$$\gamma_1 R \gamma_2 \Leftrightarrow \sup_{t\geq 0} d(\gamma_1(t),\gamma_2(t)) < \infty.$$

Set $\partial\mathbb{H}^n = {}^{\mathcal{S}}/_R$ and $\overline{\mathbb{H}^n} = \mathbb{H}^n \cup \partial\mathbb{H}^n$. We define a topology on $\overline{\mathbb{H}^n}$ in such a way that $\mathbb{H}^n$ is open and inherits its own topology, and a neighborhood of $p \in \partial\mathbb{H}^n$ is obtained in the following way: choose γ in the class of p, and let x be its starting point, let V be a neighborhood of $\dot\gamma(0)$ in the unit sphere of $T_x\mathbb{H}^n$ and let $r > 0$; then we set

$$\begin{aligned}U(\gamma,V,r) = &\{\gamma_1(t) : \gamma_1\in\mathcal{S}, \gamma_1(0) = x, \dot\gamma_1(0)\in V, t > r\}\cup\\ &\cup\{\langle\gamma_1\rangle_R : \gamma_1\in\mathcal{S}, \gamma_1(0) = x, \dot\gamma_1(0)\in V\}.\end{aligned}$$

(We omit the proof that when γ, V and r vary, $\{U(\gamma,V,r)\}$ satisfies the axioms of a fundamental system of neighborhoods of p.)

Proposition A.5.10. $\partial\mathbb{H}^n$ is homeomorphic to S^{n-1} and $\overline{\mathbb{H}^n}$ is homeomorphic to $\overline{D^n}$. Moreover if we consider the disc model $\mathbb{D}^n$ of hyperbolic space, $\overline{\mathbb{D}^n}$ is canonically identified with the closure of $\mathbb{D}^n$ as a subset of $\mathbb{R}^n$.

Proof. We shall prove the second fact, which implies the first. Given a goedesic half-line, since it is an arc of diameter or circle, it determines a unique point on ∂D^n. Moreover, two geodesic half-lines are in relation R if and only if they determine the same point on ∂D^n. Then $\overline{\mathbb{D}^n}$ is canonically identified with $\overline{D^n}$, and it is straight-forward that this identification is a homeomorphism with respect to the topology defined above. □

Remark A.5.11. Since the inversion mapping $\mathbb{D}^n$ onto $\Pi^{n,+}$ maps $\partial\mathbb{D}^n$ onto $(\mathbb{R}^{n-1}\times\{0\})\cup\{\infty\}$, this set is the natural boundary of $\Pi^{n,+}$.

Remark also that ∂D^n and $(\mathbb{R}^{n-1}\times\{0\})\cup\{\infty\}$ are two models for the sphere S^{n-1}, and hence in the disc and half-space model we can endow the boundary of $\mathbb{H}^n$ with the conformal structure of S^{n-1}.

We shall refer to the points of $\partial\mathbb{H}^n$ as the points at infinity of $\mathbb{H}^n$.

If p is a point at infinity in $\mathbb{H}^n$ we shall say a geodesic γ passes through p if p is the equivalence class of $\gamma|_{[0,\infty)}$ or $\gamma|_{(-\infty,0]}$; equivalently, we shall say that p is an endpoint of γ. It is readily verified that all geodesics have exactly two endpoints, and moreover given $p, q \in \partial\mathbf{H}^n$, $p \neq q$, there exists one and only one geodesic having endpoints p and q.

Remark that a hyperbolic subspace N can be completed to a closed submanifold of $\overline{\mathbf{H}^n}$ by adding to it all the endpoints of the geodesics it contains; these points will be called the points at infinity of N.

We shall say two geodesics in $\mathbf{H}^n$ are:

- incident, if they have a common point in $\mathbf{H}^n$,
- asymptotically parallel, if they have one common endpoint,
- ultra-parallel, if they have no intersection in $\overline{\mathbf{H}^n}$.

Remark that ultra-parallel geodesics exist even in $\mathbb{H}^2$, and this is a deep difference between the hyperbolic plane and the Euclidean one. We shall discuss now the essential properties concerning the mutual position of two hyperbolic subspaces.

Proposition A.5.12. Let $N, M \subset \mathbf{H}^n$ be hyperbolic subspaces;
(a) if N and M meet in $\partial\mathbf{H}^n$ and not in $\mathbf{H}^n$ then they have exactly one common point at infinity, and there exists no geodesic line orthogonal to both N and M;
(b) if N and M do not meet in the whole $\overline{\mathbf{H}^n}$ then there exists a geodesic γ orthogonal to both N and M, and the distance between N and M equals the length of the arc on γ lying through N and M.

Proof. (a) If N and M have two common points p and q at infinity, then both N and M contain the geodesic line having endpoints p and q, and hence they have non-trivial intersection in $\mathbb{H}^n$, which is absurd. In the half-space model we can assume the common point of N and M is ∞, *i.e.* N and M are affine vertical subspaces; $N = N_1 \times \mathbb{R}_+$, $M = M_1 \times \mathbb{R}_+$. A geodesic orthogonal to N is a circle centred at a point of N_1; but $N_1 \cap M_1 = \phi$ and hence no geodesic can be orthogonal to both N and M.

(b) It is easily checked that N and M have positive distance (otherwise they would have a common point somewhere: recall that $\overline{\mathbf{H}^n}$ is compact). Let δ be their distance, and let $\{a_n\}$ and $\{b_n\}$ be sequences in N and M respectively such that $d(a_n, b_n) \to \delta$. We can assume that these sequences converge in $\overline{\mathbf{H}^n}$; if one of the limits is a point p at infinity, the condition on the distance implies that the other limit is p too, and this is absurd since p would be a point at infinity of both N and M. It follows that $a_n \to a \in \mathbb{H}^n$ and $b_n \to b \in \mathbf{H}^n$, the convergence being with respect to the hyperbolic distance; it follows in particular that $d(a, b) = \delta$. Let γ denote the geodesic line passing through a and b; let us remark at once that the arc of γ lying through N and M is nothing but the arc from a to b, and hence it has length δ. Assume for instance that γ is not orthogonal to N in a. Choose the model $\mathbf{D}^n$. Since the hyperbolic distance is locally approximated by a multiple of the Euclidean distance (A.5.8), then we can find a point a' on γ (near a) such that its distance from N is strictly less than its distance from a (moreover we can choose a' on the side of M); this implies that the distance from N to M is strictly less than δ, and this is absurd. □

Proposition A.5.13. (1) All isometries of $\mathbf{H}^n$ extend to homeomorphisms of $\overline{\mathbf{H}^n}$, and hence they have some fixed point in $\overline{\mathbf{H}^n}$.
(2) $\mathcal{I}(\mathbf{H}^n)$ and $\mathcal{I}^+(\mathbf{H}^n)$ operate transitively on $\partial\mathbb{H}^n$ and on the set

$$\{(x, v) : x \in \mathbf{H}^n, v \in T_x\mathbf{H}^n, ds_x^2(v) = 1\}$$

where the action is defined by $f(x, v) = (f(x), d_x f(v))$.
(3) an element of $\mathcal{I}(\mathbf{H}^n)$ is uniquely determined by its trace on $\partial\mathbf{H}^n$;
(4) if M is either the disc model or the half-space model the restriction to the boundary is an isomorphism of $\mathcal{I}(M)$ onto $\mathrm{Conf}(\partial M)$.

Proof. (1) It is enough to consider the model $\mathbf{D}^n$, where this fact follows at once from the explicit determination of the isometries. The second assertion is an immediate consequence of A.5.10 and Brouwer's fixed point theorem (see [Mi2] or [Greenbl]).

(2) We consider the disc model again, where the first fact is obvious. As for the second, we only need to remark that $\mathcal{I}^+(\mathbf{D}^n)$ operates transitively on $\mathbf{D}^n$ and $\mathrm{SO}(n)$ operates transitively on S^{n-1}.

(3) Once again we consider the disc model, where this fact is obvious.

(4) By A.4.1 and A.4.2 we have to check that for $N \in \{D^n, \Pi^{n,+}\}$ the restriction to the boundary is an isomorphism of $\mathrm{Conf}(N)$ onto $\mathrm{Conf}(\partial N)$, and this is a straight-forward corollary of the explicit determination of these groups (A.3.4, A.3.5, A.3.8 and A.3.9). □

We shall give now a classification of the isometries of $\mathbf{H}^n$ with respect to their fixed points.

Proposition A.5.14. If $\phi \in \mathcal{I}(\mathbf{H}^n)$ the following mutually excluding possibilities are given:
(1) ϕ has some fixed point in $\mathbb{H}^n$;
(2) ϕ has no fixed points in $\mathbb{H}^n$, and exactly one fixed point at infinity;
(3) ϕ has no fixed points in $\mathbb{H}^n$, and exactly two fixed points at infinity.

Proof. We only have to check that if ϕ has no fixed point in $\mathbf{H}^n$ then it has at most two fixed points at infinity. In the half-space model, let us assume that ϕ has no fixed point in $\Pi^{n,+}$, and $0, \infty$ are fixed. Then ϕ can be written as

$$\phi : (y, t) \mapsto \lambda(Ay, t).$$

Since $\phi(0, 1) \neq (0, 1)$ we have $\lambda \neq 1$, and this implies that ϕ fixes only 0 and ∞. □

According to the above proposition we shall say $\phi \in \mathcal{I}(\mathbf{H}^n)$ is:

- of elliptic type if (1) occurs;
- of parabolic type if (2) occurs;
- of hyperbolic type if (3) occurs.

Remark A.5.15. If ϕ is an isometry of hyperbolic type then there exists one and only one ϕ-invariant geodesic line, whose endpoints are the fixed points at infinity for ϕ.

Now we shall specialize this classification to the case of dimensions 2 and 3 for orientation-preserving isometries.

For $n = 2$ we shall give a geometric and an algebraic classification of the isometries. Since in $\mathbb{H}^2$ the notions of length, angle, and (geodesic) line are defined, the concepts of bisecting line of an angle and axis of a segment are naturally defined too. We shall denote by $[a, b]$ the closed segment with endpoints a and b, and by (a, b, c) the angle between $[b, a]$ and $[b, c]$.

Proposition A.5.16. Let $\phi \in \mathcal{I}^+(\mathbb{H}^2) \setminus \{\mathrm{id}\}$, let x be a non-fixed point of ϕ, and let l_1 be the bisecting line of the angle $(x, \phi(x), \phi^2(x))$ and l_2 the axis of the segment $[\phi(x), \phi^2(x)]$. Then the following holds:
(1) if l_1 and l_2 are incident, ϕ is elliptic;
(2) if l_1 and l_2 are asymptotically parallel, ϕ is parabolic,
(3) if l_1 and l_2 are ultra-parallel, ϕ is hyperbolic.

Proof. Let us remark first that if ϕ is elliptic then it has only one fixed point, otherwise it would be the reflection with respect to a geodesic line which is not in $\mathcal{I}^+(\mathbb{H}^2)$. Moreover the relative position of l_1 and l_2 is invariant under the action of $\mathcal{I}(\mathbb{H}^2)$, so we can choose the fixed point(s) of ϕ in a suitable way. We carry out the proof by pictures by considering the three possible cases.
(1) ϕ elliptic; we choose $0 \in \mathbb{D}^2$ as fixed point and we obtain the situation of Fig. A.4.
(2) ϕ parabolic; we choose $\infty \in \Pi^{2,+}$ as fixed point and we obtain the situation of Fig. A.5.
(3) ϕ hyperbolic; we choose $0, \infty \in \Pi^{2,+}$ as fixed points and we obtain the situation of Fig. A.6.

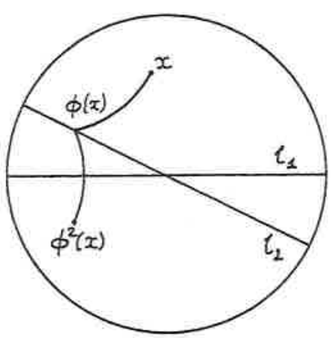

Fig. A.4. Geometric classification of isometries in dimension two: elliptic case

□

According to A.4.2 and A.3.3 every orientation-preserving isometry of $\Pi^{2,+}$ is represented by a 2×2 real matrix with determinant 1, and it is

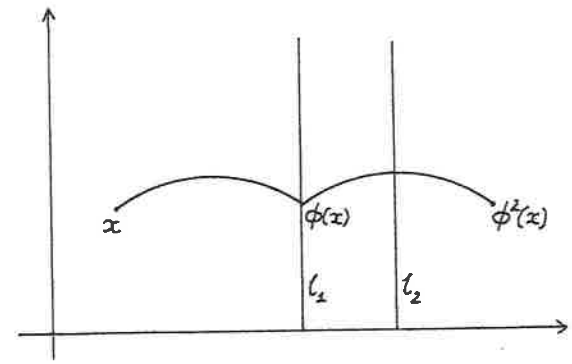

Fig. A.5. Geometric classification of isometries in dimension two: parabolic case

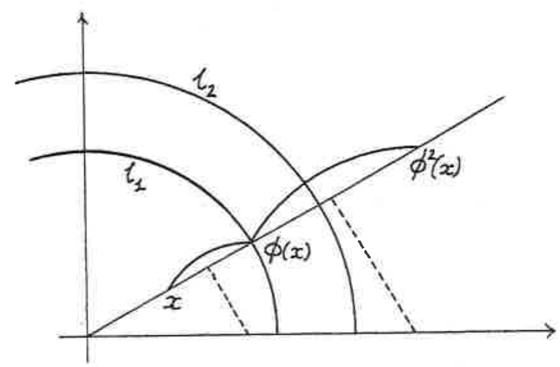

Fig. A.6. Geometric classification of isometries in dimension two: hyperbolic case

easily checked that two matrices A and B represent the same isometry if and only if $A = \pm B$. We shall denote by tr the trace of a matrix.

Proposition A.5.17. Let $\phi \in \mathcal{I}^+(\Pi^{2,+}) \setminus \{\mathrm{id}\}$ be represented by a matrix $A \in \mathrm{Sl}(2, \mathbb{R})$; then
(1) if $|tr(A)| < 2$, ϕ is elliptic;
(2) if $|tr(A)| = 2$, ϕ is parabolic;
(3) if $|tr(A)| > 2$, ϕ is hyperbolic.

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We recall that $\phi(z) = {}^{(az+b)}/_{(cz+d)}$. If $c = 0$ then ∞ is a fixed point, $\phi(z) = a^2 z + ab$ and $tr(A) = a + {}^1/_a$. If $a = \pm 1$ then $b \neq 0$ (otherwise $\phi = \mathrm{id}$, which is absurd). It follows that

$$tr(A) = \pm 2 \;\Leftrightarrow\; a = \pm 1 \;\Leftrightarrow\; \phi \text{ parabolic}$$
$$tr(A) > 2 \;\Leftrightarrow\; a \neq \pm 1 \;\Leftrightarrow\; \phi \text{ hyperbolic.}$$

If $c \neq 0$ we consider the equation

$$\phi(z) = z \iff cz^2 + (d-a)z - b = 0$$

having discriminant $\Delta = (d-a)^2 + 4bc = tr(A)^2 - 4$, and the above cases are easily discussed. □

Now we turn to the three-dimensional case. By A.4.2 and A.5.13, the restriction of an orientation-preserving isometry of $\mathbb{H}^3$ to the boundary is a conformal diffeomorphism of S^2, and conversely every element of $\mathrm{Conf}^+(S^2)$ can be extended in a unique way to an element of $\mathcal{I}^+(\mathbb{H}^3)$. Moreover by A.3.3

$$\mathrm{Conf}^+(S^2) \cong {\mathrm{Sl}(2,\mathbb{C})}/_{\{\pm I\}}$$

and hence

$$\mathcal{I}^+(\mathbb{H}^3) \cong {\mathrm{Sl}(2,\mathbb{C})}/_{\{\pm I\}}.$$

Proposition A.5.18. Let $\phi \in \mathcal{I}^+(\mathbb{H}^3) \setminus \{\mathrm{id}\}$ be represented by a matrix $A \in \mathrm{Sl}(2,\mathbb{C})$. Then
(1) if $tr(A) \in \mathbb{R}$, $|tr(A)| < 2$, ϕ is elliptic;
(2) if $tr(A) = \pm 2$, ϕ is parabolic;
(3) if $tr(A) \notin \mathbb{R}$ or $tr(A) \in \mathbb{R}$, $|tr(A)| > 2$, ϕ is hyperbolic.

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc = 1$; since the equation

$$\frac{az+b}{cz+d} = z$$

has some solution in $\mathbb{C} \cup \{\infty\}$, then A has a fixed point in $\mathbb{C} \cup \{\infty\}$. Moreover we have that:
- $\mathrm{Sl}(2,\mathbb{C})$ operates transitively on $\mathbb{C} \cup \{\infty\}$;
- $tr(B^{-1}AB) = tr(A)\ \ \forall B \in \mathrm{Sl}(2,\mathbb{C})$;
- ϕ and $\psi^{-1}\phi\psi$ are of the same type $\forall \psi \in \mathcal{I}(\mathbb{H}^3)$.

Therefore we can assume A fixes ∞, *i.e.* $c = 0$, $d = {}^1/_a$, $tr(A) = a + {}^1/_a$, $A(z) = a^2 z + ab$. The isometry ϕ of $\Pi^{3,+}$ (characterized by the fact that it extends A) is then given by

$$\phi(z,t) = (a^2 z + ab, |a|^2 t).$$

As in A.5.17, if $a = \pm 1$ then $b \neq 0$, and therefore we have:
- ϕ elliptic $\iff |a| = 1, a \neq \pm 1$,
- ϕ parabolic $\iff a = \pm 1$,
- ϕ hyperbolic $\iff |a| \neq 1$,

and the conclusion follows easily. □

We introduce now a new geometric notion: the horosphere. Given $p \in \partial\mathbb{H}^n$ we shall say a closed hypersurface N in $\mathbb{H}^n$ is a horosphere centred at p if N is orthogonal to all geodesic lines with endpoint p.

Proposition A.5.19. Given $p \in \partial\mathbb{H}^n$, $\mathbb{H}^n$ is the disjoint union of the horospheres centred at p. These horospheres inherit from $\mathbb{H}^n$ the Riemannian structure of $\mathbb{R}^{n-1}$.

Proof. The notions we are considering are invariant under isometries, and hence we shall assume in the half-space model that $p = \infty$. Since geodesics with endpoint ∞ are given by vertical lines $\{x_0\} \times \mathbb{R}_+$, horospheres centred at ∞ are horizontal hyperplanes $\mathbb{R}^{n-1} \times \{t_0\}$ (and conversely such a hyperplane is a horosphere centred at ∞). The first assertion is proved. As for the second, using A.4.3, we only have to remark that a positive multiple of the standard Riemannian metric of $\mathbb{R}^{n-1}$ is equivalent to it. □

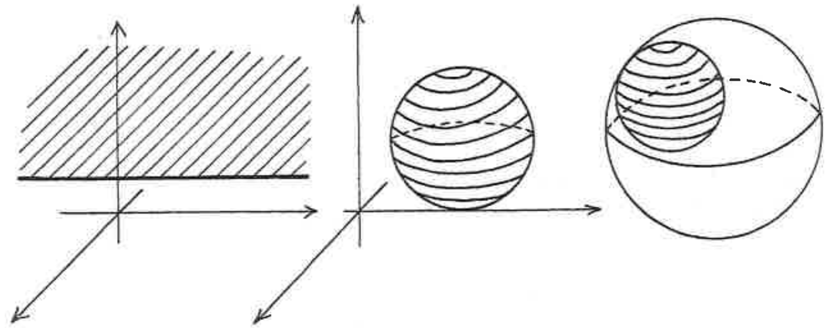

Fig. A.7. Horospheres in the three-dimensional disc and half-space models

Remark A.5.20. A horosphere in $\mathbb{H}^n$ inherits the Riemannian structure of $\mathbb{R}^{n-1}$, but not the metric space structure (that is, the distance of $\mathbb{H}^n$ restricted to a horosphere is not Euclidean). The reason is that if we integrate the metric (in order to obtain the distance) before considering the restriction to the horosphere, we use geodesics which are not contained in the horosphere, and hence if we perform the two operations in the opposite order we obtain a different result.

According to the above characterization a horosphere centred at $p \in \partial\mathbb{H}^n$ divides $\mathbb{H}^n$ into two connected regions homeomorphic to n-balls: we shall call the one meeting $\partial\mathbb{H}^n$ in p a horoball centred at p.

The next result provides an alternative proof of the fact that $\mathbb{H}^n$ is complete (we have already proved this in A.5.2).

Proposition A.5.21. A hyperbolic ball in $\mathbb{D}^n$ (or $\Pi^{n,+}$) is a Euclidean ball with different centre and radius, whose closure is compact in $\mathbb{D}^n$ (or $\Pi^{n,+}$).

Proof. By A.5.8 a ball of radius r centred at 0 in $\mathbb{D}^n$ is a Euclidean ball of centre 0 and radius $\mathrm{th}({}^r/_2) < 1$ and hence its closure is compact in $\mathbb{D}^n$ (recall that the hyperbolic distance induces the standard topology on D^n). Since inversions with respect to spheres orthogonal to $\partial\mathbb{D}^n$ and elements of $O(n)$ are homeomorphisms of $\overline{\mathbb{D}^n}$ and they map balls into balls, the proposition holds for every ball in $\mathbb{D}^n$. Moreover the mapping used for the definition of $\mathbb{H}^{n,+}$ is an inversion too, and the proposition is proved. □

We give now a description of the "banana" neighborhoods of geodesics in $\mathbb{H}^n$. Given a geodesic γ in $\mathbb{H}^n$ and $\varepsilon > 0$ we consider the closed tubular ε-neighborhood of γ:

$$N_\varepsilon(\gamma) = \{x \in \mathbb{H}^n : d(x,\gamma) \leq \varepsilon\};$$

if in $\mathbb{H}^{n,+}$ we have $\gamma = \{0\} \times \mathbb{R}_+$ then $N_\varepsilon(\gamma)$ is invariant under dilations (as γ is) and it is easily checked (by A.5.8) that $N_\varepsilon(\gamma)$ is the infinite cone based on a horizontal closed $(n-1)$-disc. Now, an immediate argument based on the properties of inversions proves that the general shape of $N_\varepsilon(\gamma)$ in $\mathbb{D}^n$ and $\mathbb{H}^{n,+}$ is the one described in Fig. A.8.

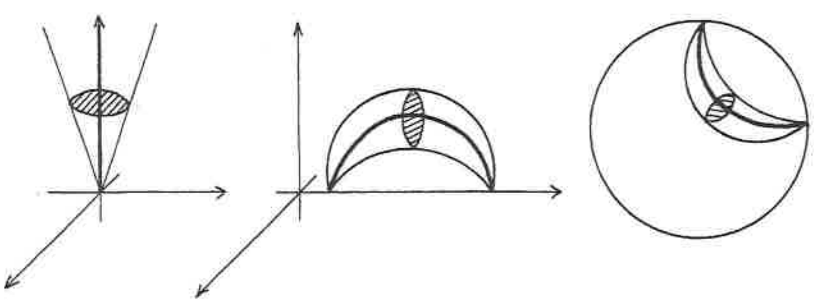

Fig. A.8. Neighborhoods of geodesics in the three-dimensional disc and half-space

Remark A.5.22. As a conclusion of the section we point out some peculiarities of the the projective model for $\mathbb{H}^n$ we introduced at the beginning. The first feature is the following: if we canonically identify it with the unit disc D^n we have that the conformal structure D^n inherits from the hyperbolic structure is not equivalent to the usual one. Moreover it is not difficult to check that the geodesic subspaces are given in this model by the intersections of D^n with the affine subspaces of $\mathbb{R}^n$; we deduce from this for instance that a geodesic polyhedron in this model is a Euclidean geodesic polyhedron. This fact is sometimes useful as some arguments applying to Euclidean geodesic polyhedra generalize to the hyperbolic case (provided they do not involve the notion of measure of an angle).

A.6 Curvature of Hyperbolic Space

We shall prove that in every point of $\mathbb{H}^n$ the sectional curvature of $\mathbb{H}^n$ with respect to any section is -1. Before proving this in detail we give a result on the strict convexity of the distance function, which expresses qualitatively the fact that $\mathbb{H}^n$ is negatively curved. If $x, y \in \mathbb{H}^n$ we shall denote by $(x+y)/2$ the middle point of the geodesic arc joining x and y.

Proposition A.6.1. Let γ_1, γ_2 be closed geodesic arcs in $\mathbb{H}^n$ having in common at most one endpoint and such that they are not arcs of the same maximal geodesic; let $x, x' \in \gamma_1$ and $y, y' \in \gamma_2$ with $x \neq x'$ and $y \neq y'$ and set $p = (x+x')/2$, $q = (y+y')/2$; then

$$2d(p,q) < d(x,y) + d(x',y').$$

Proof. Since $p \in \gamma_1$ and $q \in \gamma_2$ we cannot have $p = q$, otherwise this point would be an endpoint of γ_1, whence $x = x' = p$, which is absurd. Let δ be the maximal geodesic passing through p and q; if both x and y belong to δ then γ_1 and γ_2 are arcs of δ, which is absurd. Let us assume that $x \notin \delta$ (whence $x' \notin \delta$). Let σ be the symmetry with respect to p, *i.e.* the isometry of $\mathbb{H}^n$ characterized by the relation $p = (w + \sigma(w))/2$ $\forall w \in \mathbb{H}^n$, and similarly let ρ be the symmetry with respect to q. Set $\tau = \rho \circ \sigma$ and $z = \tau(x) = \rho(x')$, $z' = \tau(x') = \rho(x)$, $r = \tau(p) = \rho(p)$; then we have

$$\begin{aligned} 2d(p,q) &= d(p,r) \\ d(y',z') &= d(\rho(y),\rho(x)) = d(y,x) \\ d(x',z') &\leq d(x',y') + d(y',z'). \end{aligned}$$

So it is enough to prove that $d(p,r) < d(x',z')$, *i.e.*

$$d(p,\tau(p)) < d(x',\tau(x')).$$

Let us assume that in the half-space model $\delta = \{0\} \times \mathbb{R}_+$; since τ is the product of two symmetries with respect to different points of δ, it is easily calculated that τ is a dilation of coefficient $\lambda \neq 1$. Moreover p has the form $(0, t_1)$ and x' has the form (a, t_2) with $a \in \mathbb{R}^{n-1} \setminus \{0\}$; by A.5.8 it is easily verified that

$$d\big((0,t_1),(0,\lambda t_1)\big) < d\big((a,t_2),(\lambda a,\lambda t_2)\big).$$ □

The situation considered in the next result was not included in A.6.1 for technical reasons and in order to emphasize it with a specific statement. We shall say three points in $\mathbb{H}^n$ are non-aligned if each geodesic in $\mathbb{H}^n$ contains at most two of them.

Corollary A.6.2. Let x, x', y be non-aligned points of $\mathbb{H}^n$ and define p as $(x+x')/2$; then

$$2d(p,y) < d(x,y) + d(x',y).$$

Proof. Set $y' = q = y$. Since x does not belong to the geodesic passing through p and q, the above proof works in this case too. □

Remark A.6.3. With the same symbols as in E.6.1 we have in $\mathbb{R}^n$ the same inequality

$$2d(p,q) \leq d(x,y) + d(x',y')$$

but we do have non-trivial cases when equality holds; moreover in S^n (endowed with the metric it inherits from $\mathbb{R}^{n+1}$) we have cases when the opposite inequality

$$2d(p,q) > d(x,y) + d(x',y')$$

holds. These facts express qualitatively the fact that $\mathbb{H}^n$, $\mathbb{R}^n$ and S^n have different curvature.

We are now going to prove the assertion about the sectional curvatures of $\mathbb{H}^n$. We recall that the curvature at a point of an oriented Riemannian surface can be defined in the following equivalent ways (see for instance [Boo], [Ga-Hu-La], [Ko-No] or [Sp]):

– via the definition of the Levi-Civita connection and the Riemann tensor associated to it; in particular, if Ω is a domain in $\mathbb{R}^2$, $\alpha : \Omega \to \mathbb{R}_+$ is a C^∞ function and a differential metric on Ω is defined by $ds_x^2(v) = \alpha(x)^2 \cdot \|v\|^2$, this procedure yields the following expression of the curvature at a point x:

$$k(x) = -\frac{1}{\alpha(x)^2} \cdot (\Delta \log \alpha)(x)$$

where Δ denotes the Laplace operator.

– via the definition of parallel transport and of a function ϕ associating to each pre-compact domain D with smooth boundary the number

$$\phi(D) = \sphericalangle(v, P_{\partial D}(v)),$$

where $P_{\partial D}(v)$ denotes the parallel transport along ∂D in the positive direction of a vector v tangent to a point of ∂D, and $\sphericalangle(v,w)$ denotes the measure with sign of the angle between v and w: it is shown that for a suitable function k on the surface

$$\phi(D) = \int_D k(x)dm(x)$$

where $dm(x)$ denotes the element of area at x.

The sectional curvature in a point x of a Riemannian manifold M with respect to a 2-subspace $V \subset T_xM$ (called a section at x) is defined as the curvature at x of the oriented Riemannian surface obtained as the image under the exponential mapping of a suitably small neighborhood of 0 in V.

Lemma A.6.4. The sectional curvature of $\mathbb{H}^n$ in a point x with respect to a section $V \subset T_x\mathbb{H}^n$ is independent of V, x and n.

Proof. Independence of V and x follows at once from the isometry-invariance of the curvature and from the fact that $\mathcal{I}(\mathbb{H}^n)$ operates transitively on the pairs

x, V. As for the last assertion we only need to recall that by A.5.7 everything reduces to the case $n = 2$: in fact the image under the exponential mapping of a section at any point of $\mathbb{H}^n$ is a hyperbolic 2-subspace. □

Lemma A.6.5. Let T be a geodesic triangle in $\mathbb{H}^2$ with inner angles α, β, γ, let $m(T)$ denote its measure and let $\phi(T)$ be defined as above; then

(a) $m(T) = \pi - (\alpha + \beta + \gamma)$;

(b) $\phi(T) = \alpha + \beta + \gamma - \pi$.

Proof. (a) Let us consider the half-space model. We start by computing the area of a geodesic triangle Δ having a vertex at ∞ and inner angles $\alpha, \beta, 0$.

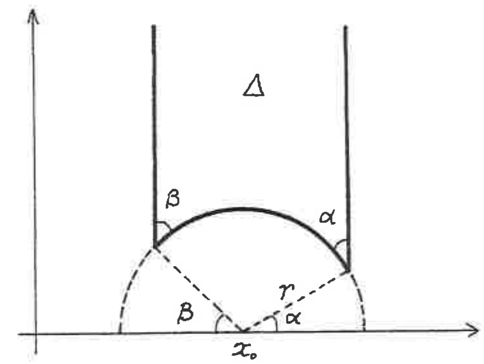

Fig. A.9. Computation of the area of a geodesic triangle in hyperbolic two-space; the case of a vertex at infinity

We have that $m(\Delta) = \int_\Delta {dx\,dy}/{y^2}$. Since

$$\Delta = \{(x_0 + r\cos\vartheta, y) : \alpha \leq \vartheta \leq \pi - \beta,\ y \geq r\sin\vartheta\}$$

we have that

$$m(\Delta) = \int_\alpha^{\pi-\beta} r\sin\vartheta d\vartheta \int_{r\sin\vartheta}^{\infty} \frac{dy}{y^2} = \int_\alpha^{\pi-\beta} d\vartheta = \pi - \alpha - \beta.$$

Now, for the general case, we remark that the area of T can be expressed as the algebraic sum of the areas of three geodesic triangles having a vertex at ∞, and the conclusion follows quite easily by considering the different possibilities. For instance in the situation of Fig. A.10 we consider the triangles:

Δ_1 with vertices x, y, ∞ and inner angles $\alpha_1, \beta_1, 0$;

Δ_2 with vertices x, z, ∞ and inner angles $\alpha_2, \gamma_1, 0$;

Δ_3 with vertices y, z, ∞ and inner angles $\beta_2, \gamma_2, 0$;

as suggested in Fig. A.11.

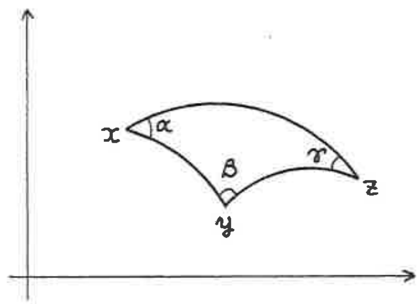

Fig. A.10. Computation of the area of a geodesic triangle in hyperbolic two-space; general case

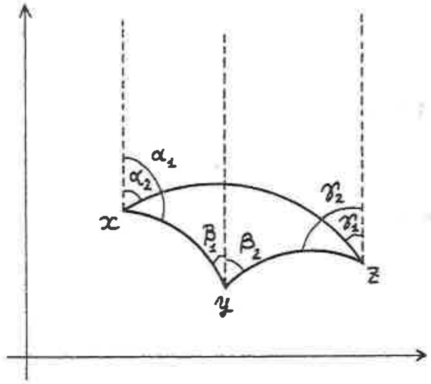

Fig. A.11. Computation of the area of a geodesic triangle in hyperbolic two-space; how to divide a general triangle

We have the obvious relations

$$\alpha_1 = \alpha + \alpha_2 \qquad \beta = \beta_1 + \beta_2 \qquad \gamma_2 = \gamma + \gamma_1$$

and hence

$$\begin{aligned} m(T) &= m(\Delta_1) + m(\Delta_3) - m(\Delta_2) = \\ &= \pi - \alpha_1 - \beta_1 + \pi - \beta_2 - \gamma_2 - (\pi - \alpha_2 - \gamma_1) = \\ &= \pi - \alpha - \beta - \gamma. \end{aligned}$$

(b) In $\mathbb{H}^{2,+}$ we can assume that one of the sides of T is vertical; the proof is contained in Fig. A.12.

□

Remark A.6.6. The above result holds also if T is assumed to be a geodesic triangle in $\overline{\mathbb{H}^2}$, *i.e.* the vertices are allowed to be points at infinity. (The proof works just as above.) In particular, the area of any geodesic triangle having all vertices at infinity is π. We shall use this fact while discussing the rigidity theorem (Sect. C.2).

Theorem A.6.7. All sectional curvatures of $\mathbb{H}^n$ are -1.

Proof. We present two proofs, according to the two possible definitions of the sectional curvature given above. In both cases, recalling A.6.3, we consider only $n = 2$.

– $\mathbb{D}^2$ is the Riemannian surface associated to the function $\alpha : D^2 \to \mathbb{R}_+$, $\alpha(x) = {}^2/_{(1-\|x\|^2)}$, and it easily checked that

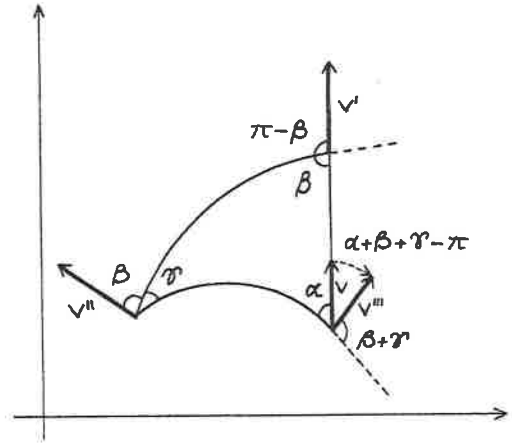

Fig. A.12. Calculation of the parallel transport along the boundary of a geodesic triangle in hyperbolic two-space

$$-\frac{1}{\alpha(0)^2} \cdot (\Delta \log \alpha)(0) = -1.$$

– For $x \in \mathbb{H}^2$ we can consider a sequence of geodesic triangles $\{T_n\}$ such that $x \in T_n$ and

$$\lim_{n\to\infty} diam(T_n) = 0.$$

Then, by A.6.5,

$$k(x) = \lim_{n\to\infty} {}^{\phi(T_n)}/_{m(T_n)} = -1.$$

□

Remark A.6.8. Lemma A.6.5 (a) provides another evidence of the fact that a unit of measure is intrinsically defined in $\mathbb{H}^n$ (compare A.4.5).

The following determination of the curvature of S^n and $\mathbb{R}^n$ is easily obtained:

Theorem A.6.9. All sectional curvatures of S^n are 1 and all sectional curvatures of $\mathbb{R}^n$ are 0.

We conclude this chapter with the construction of a surface in $\mathbb{R}^3$ having constant curvature -1 (with respect to the Riemannian metric induced from $\mathbb{R}^3$). Let us consider the Euclidean plane $\mathbb{R}^2 = \mathbb{R}_x \times \mathbb{R}_y$; we shall call tractrix a curve α in the open first quadrant such that the distance between $\alpha(t)$ and the intersection of the y-axis with the tangent line to α at $\alpha(t)$ is identically 1. It is quite easily verified that the tractrix exists and is unique up to change of parameter, and in particular it is the graphic of the function

$$y(x) = \int_x^1 \sqrt{1/t^2 - 1}\, dt \qquad x \in (0,1).$$

(The integral can be explicitly computed, but it does not say very much.)

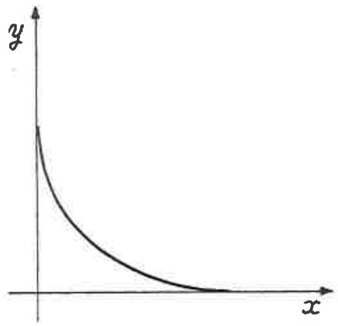

Fig. A.13. The tractrix

In $\mathbb{R}^3$ we shall call pseudo-sphere the surface generated by the rotation of the tractrix around the y-axis.

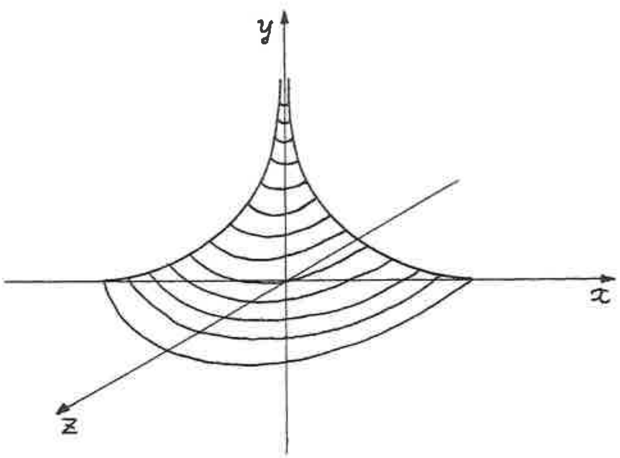

Fig. A.14. The pseudo-sphere

The pseudo-sphere in a neighborhood of a point $(x_0, y(x_0), 0)$ is the graphic of the function

$$(x,z) \mapsto \sqrt{y(x)^2 - z^2}$$

and a straight-forward application of the general formulae for the curvature of a graphic yields:

Proposition A.6.10. The curvature of the pseudo-sphere is identically -1.

Of course the pseudo-sphere is not a complete Riemannian manifold; the following result implies that this fact cannot be avoided (see [DC]):

Proposition A.6.11. No complete surface in $\mathbb{R}^3$ can have strictly negative curvature everywhere.